Available online at http://jprm.sms.edu.pk/
Journal of Prime Research in Mathematics, 19(1) (2023), 27–33

Journal of Prime Research in Mathematics

On the solutions of $2^x + 2^y = z^2$ in the Fibonacci and Lucas numbers

Hayder R. Hashim

Faculty of Computer Science and Mathematics, University of Kufa, P.O. Box 21, 54001 Al Najaf, Iraq.

Abstract

Consider the Diophantine equation $2^x + 2^y = z^2$, where x, y and z are nonnegative integers. As this equation has infinitely many solutions, in this paper we study its solutions in case where the unknowns represent Fibonacci and/or Lucas numbers. In other words, we completely resolve the equation in case of $(x, y, z) \in \{(F_i, F_j, F_k), (F_i, F_j, L_k), (L_i, L_j, L_k), (L_i, L_j, F_k), (F_i, L_j, L_k), (F_i, L_j, F_k)\}$ with $i, j, k \geq 1$ and F_n and L_n denote the general terms of Fibonacci and Lucas numbers, respectively.

Keywords: Diophantine equations, exponential Diophantine equations, Fibonacci sequence, Lucas sequence.
2010 MSC: 11D61, 11B39.

1. Introduction and Preliminaries

Diophantine equations of the form

$$a^x + b^y = c^z \tag{1.1}$$

have been widely studied by many authors in certain cases. For instance, in 1958 Nagell [9] determined all the positive integer solutions of equation (1.1) under the condition that a, b, c are prime numbers such that $\max\{a, b, c\} \leq 7$ and $a > b$. Starting from 1959 several authors obtained the solutions of equation (1.1) in case of $11 \leq \max\{a, b, c\} \leq 23$ such that Makowski [8], Hadano [4], Uchiyama [18], Sun and Zhou [16] and Yang [19]. In 1988, Cao [2] gave all the sixty solutions in total for equation (1.1) when $29 \leq \max\{a, b, c\} \leq 97$. More precisely, he proved that if $\max\{a, b, c\} > 13$, then (1.1) has at most one solution in positive integers. Furthermore, in 1999 [3] he obtained a more general result. In fact, many equations of the form (1.1) have been considered. For example, in 2002 Sándor [13] published a booklet that collects some of his papers dealing with such equations, e.g. $3^x + 3^y = 6^z$ and $4^x + 18^y = 22^z$. Later in 2007, Acu [1] investigated the solutions of the equation

$$2^x + 5^y = z^2.$$

*Corresponding author
Email address: `hayderr.almuswi@uokufa.edu.iq` (Hayder R. Hashim)

Received : 30 September 2022*; Accepted:* 10 March 2023*; Published Online:* 8 June 2023.

He indeed proved that this equation has only two solutions in nonnegative integers, namely $(x, y, z) = (3, 0, 3)$ and $(2, 1, 3)$. Since then, there have been increasing interests in studying the solutions of a general form of the latter equation; that is

$$p^x + q^y = z^2,$$

see e.g. [11] and the references given there. One of the interesting results was given by Suvarnamani [17] in which he studied the solutions of the Diophantine equation

$$2^x + q^y = z^2, \tag{1.2}$$

where the unknowns x, y and z are nonnegative integers and q is a prime number. In fact, he showed that if $q = 2$ then equation (1.2) has infinitely many solutions. More precisely, the solutions are given by

$$(x, y, z) \in \{(2s-1, 2s-1, 2^s), (2r+3, 2r, 3\cdot 2^r), (2r, 2r+3, 3\cdot 2^r)\},$$

where s and r are positive and nonnegative integers, respectively. On the other hand, Diophantine equations connected to linear recurrence sequences have been widely studied by many mathematicians, see e.g. [5], [6], [10], [14] and the references given there.

Therefore, in this paper we answer the question of whether or not the Diophantine equation

$$2^x + 2^y = z^2 \tag{1.3}$$

has infinitely many solutions if x, y and z are Fibonacci and/or Lucas numbers. In other words, we investigate the solutions of each of the following Diophantine equations:

$$2^{F_i} + 2^{F_j} = F_k^2, \tag{1.4}$$
$$2^{L_i} + 2^{L_j} = L_k^2, \tag{1.5}$$
$$2^{L_i} + 2^{L_j} = F_k^2, \tag{1.6}$$
$$2^{F_i} + 2^{L_j} = F_k^2, \tag{1.7}$$
$$2^{F_i} + 2^{F_j} = L_k^2, \tag{1.8}$$
$$2^{F_i} + 2^{L_j} = L_k^2, \tag{1.9}$$

where the indices i, j and k are positive integers, and F_n and L_n respectively represent the nth terms of the Fibonacci and Lucas sequences, that are defined by the following recurrence relations:

$$F_0 = 0, F_1 = 1, \quad F_n = F_{n-1} + F_{n-2} \quad \text{for} \quad n \geq 2,$$

and

$$L_0 = 2, L_1 = 1, \quad L_n = L_{n-1} + L_{n-2} \quad \text{for} \quad n \geq 2.$$

It is known that the characteristics polynomial of these sequences is defined by

$$x^2 - x - 1 = 0,$$

whose roots are

$$\alpha = \frac{1+\sqrt{5}}{2} \quad \text{and} \quad \beta = \frac{1-\sqrt{5}}{2},$$

which imply that $\beta = \frac{-1}{\alpha}$. Hence, the Benit's formulas of the Fibonacci and Lucas sequences are defined by

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \qquad \text{and} \qquad L_n = \alpha^n + \beta^n \tag{1.10}$$

for all $n \geq 0$. Moreover, one can easily prove that

$$\alpha^{n-2} \leq F_n \leq \alpha^{n-1} \qquad \text{for all} \qquad n \geq 1, \tag{1.11}$$

and

$$\alpha^{n-1} \leq L_n \leq \alpha^{n+1} \qquad \text{for all} \qquad n \geq 1. \tag{1.12}$$

These sequences are connected by the identity relationship between the Fibonacci and Lucas sequences

$$L_n^2 = 5F_n^2 \pm 1. \tag{1.13}$$

In fact, there are many results and identities related to these sequences, and for more details see e.g. [7] or [12]. Later in this paper, we show an interesting result concerning the solutions of the above problems presented in equations (1.4)–(1.9) where the unknowns are Fibonacci and/or Lucas numbers; that is these equations have finitely many solutions, that are completely determined in Section 2.

Remark 1.1. Our argument of attacking the above problems represented in equations (1.4)–(1.9) is based on providing an upper bound for the minimum of the indices. So, in order to completely resolve each of the equations, we have to determine all the possible values of i, j and k in which the studied equation is satisfied. In other words, we have to determine the solutions of each equation at the cases: $i \leq j \leq k, j \leq i \leq k, k \leq i \leq j, k \leq j \leq i, i \leq k \leq j$ and $j \leq k \leq i$. Therefore, in order to eliminate some of these cases, we introduce the following lemma:

Lemma 1.2. *If any of the equations* (1.4)–(1.9) *is satisfied at arbitrary values of* $i, j, k \geq 9$, *then* $i, j < k$.

Proof. Note that the proof of this lemma can be conducted by a contradiction. Let's first consider equation (1.4) that holds at some of the integers $i, j, k \geq 9$ and assume for a contradiction that $i \geq k$ or $j \geq k$. Thus, equation (1.4) leads to

$$2^{F_k} < 2^{34} + 2^{F_k} \leq 2^{F_i} + 2^{F_j} = F_k^2, \tag{1.14}$$

which is false since one can easily use induction to prove that

$$2^{F_k} > F_k^2 \tag{1.15}$$

for all $k \geq 9$. That can be preformed as follows:

- `Base step:` If $k = 9$, then inequality (1.15) is clearly satisfied as $2^{34} > 34^2$.
- `Inductive step:` Suppose that inequality (1.15) holds for all $k \geq 9$. Therefore,

$$2^{F_{k+1}} = 2^{F_k}2^{F_{k-1}} > 2^{21}F_k^2 > 4F_k^2 + 2097148F_k^2 > 4F_k^2 + 2424303088 > 4F_k^2 > F_{k+1}^2$$

  as $k \geq 9$, since $F_{k+1}^2 = F_k^2 + 2F_{k-1}F_k + F_{k-1}^2 < 4F_k^2$.

That proves inequality (1.15) and leads to the contradiction of inequality (1.14). Hence, the statement of Lemma 1.2 in case of equation (1.4) is proved.

Similarly, by applying the same approach on equations (1.5), (1.6) and (1.8) with assuming for a contradiction that $i \geq k$ or $j \geq k$ such that $k \geq 9$, we respectively obtain the following inequalities:

$$2^{L_k} < 2^{76} + 2^{L_k} \leq 2^{L_i} + 2^{L_j} = L_k^2, \tag{1.16}$$

$$2^{L_k} < 2^{76} + 2^{L_k} \leq 2^{L_i} + 2^{L_j} = F_k^2 \tag{1.17}$$

and

$$2^{F_k} < 2^{34} + 2^{F_k} \leq 2^{F_i} + 2^{F_j} = L_k^2. \tag{1.18}$$

Regarding to inequality (1.16), one can easily use induction (as shown above with inequality (1.15), so we omit the detail of the proof) to show that

$$2^{L_k} > L_k^2 \quad \text{for all} \quad k \geq 9. \tag{1.19}$$

Again, we obtain a contradiction. Note that as $L_k > F_k$ for all $k \geq 9$, then inequality (1.17) implies that

$$2^{F_k} < F_k^2$$

which is not true as shown above in (1.14) and (1.15). Similarly, inequality (1.18) is not true for all $k \geq 9$, and that can be shown by using induction to prove that

$$2^{F_k} > L_k^2 \quad \text{for all} \quad k \geq 9. \tag{1.20}$$

For the simplicity of the base step, we only start with the inductive hypothesis, for which the above inequality is satisfied, to show that

$$2^{F_{k+1}} = 2^{F_k} 2^{F_{k-1}} > 2^{21} L_k^2 > L_{k+1}^2$$

for all $k \geq 9$, and that shows the truthiness of inequality (1.20) and falseness of inequality (1.18). Hence, we get a contradiction. Therefore, the statement of Lemma 1.2 regarding to equations (1.5)–(1.6) and (1.8) is also proved.

It remains to prove the statement of Lemma 1.2 regarding to equations (1.7) and (1.9) as $i, j, k \geq 9$. In a similar way, we use the same approach used earlier by assuming that the equations are satisfied at some of i, j and k with $i \geq k$ or $j \geq k$. First, equation (1.7) gives that

$$2^{L_k} < 2^{34} + 2^{L_k} \leq 2^{F_i} + 2^{L_j} = F_k^2 \tag{1.21}$$

or

$$2^{F_k} < 2^{76} + 2^{F_k} \leq 2^{F_i} + 2^{L_j} = F_k^2 \tag{1.22}$$

as $j \geq k$ or $i \geq k$, respectively. Again, from the proof of (1.15) we can easily conclude that the inequalities (1.21) and (1.22) are false as $k \geq 9$. Here is also a contradiction. Similarly, equation (1.9) leads to the following inequalities:

$$2^{L_k} < 2^{34} + 2^{L_k} \leq 2^{F_i} + 2^{L_j} = L_k^2,$$
$$2^{F_k} < 2^{76} + 2^{F_k} \leq 2^{F_i} + 2^{L_j} = L_k^2.$$

Indeed, from the proofs of inequality (1.19) and inequality (1.20), we respectively conclude the latter two inequalities are false for all $k \geq 9$. Similarly, we obtain a contradiction, and that completes the proof of Lemma 1.2. □

Remark 1.3. From Lemma 1.2, we eliminated four cases of the ones mentioned in Remark 1.1, i.e. $k \leq i \leq j, k \leq j \leq i, i \leq k \leq j$ and $j \leq k \leq i$ where $i, j, k \geq 9$. Therefore, it only remains to study the solutions of equations (1.4)–(1.9) at the cases $9 \leq i \leq j \leq k$ and $9 \leq j \leq i \leq k$. Furthermore, the solutions of each of these equations with $1 \leq i, j, k \leq 8$ at all of these cases can be determined easily using any mathematical software, e.g. SageMath [15]. In addition to that, later in the proof of our results we mainly fix the condition that $9 \leq i \leq j < k$ in each of the equations (1.4)–(1.6) and (1.8) to determine the corresponding set of the solutions $\{(i, j, k)\}$ since the set of solutions related to the remaining case can be obtained easily by permuting the component i with j, namely it is represented by the set $\{(j, i, k)\}$. However, in case of equations (1.7) and (1.9) we consider both of the cases $9 \leq i \leq j \leq k$ and $9 \leq j \leq i \leq k$.

2. Main results

Lemma 2.1. *If the triple (i, j, k) with $i, j, k \geq 9$ is a solution of any of the equations (1.4)–(1.9), then $i + j < k$ and $2j < k$.*

Proof. As followed in the proof of Lemma 1.2, the proof of this lemma can be achieved by a contradiction. Here, we may consider e.g. equation (1.5) that holds at the triple (i, j, k) with $i, j, k \geq 9$ such that $i \leq j < k$ and assume for a contradiction that $i + j \geq k$ or $2j \geq k$. Hence, we obtain that

$$2^{L_j} < 2^{76} + 2^{L_j} \leq 2^{L_i} + 2^{L_j} = L_k^2 \leq L_{2j}^2 \tag{2.1}$$

as $i \leq j$ with $i, j \geq 9$. Indeed, this is not true, and we can use induction to prove that by showing

$$2^{L_j} > L_{2j}^2 \tag{2.2}$$

for all $j \geq 9$. We may start with the inductive step by assuming the latter inequality holds for all of $j \geq 9$. Thus,

$$2^{L_{j+1}} = 2^{L_j} 2^{L_{j-1}} > 2^{47} L_{2j}^2 > 9L_{2j}^2 > L_{2j+2}^2, \tag{2.3}$$

since $L_{2j+2}^2 = 4L_{2j}^2 + 4L_{2j}L_{2j-1} + L_{2j-1}^2 < 9L_{2j}^2$. This completes the proof of inequality (2.2) and leads to the contradiction of (2.1). Therefore, this implies that $i + j < k$ and $2j < k$. Next, we similarly deal with equations (1.4), (1.6) and (1.8), which give the following inequalities:

$$2^{F_j} < 2^{34} + 2^{F_j} \leq 2^{F_i} + 2^{F_j} = F_k^2 \leq F_{2j}^2,$$
$$2^{L_j} < 2^{76} + 2^{L_j} \leq 2^{L_i} + 2^{L_j} = F_k^2 \leq F_{2j}^2$$

and

$$2^{F_j} < 2^{34} + 2^{F_j} \leq 2^{F_i} + 2^{F_j} = L_k^2 \leq L_{2j}^2,$$

respectively. As done above, one can easily get a contradiction by showing each of the latter three inequalities are not true for all $j \geq 9$. Therefore, we omit the detail of computations. Again, we here get that $i + j < k$ and $2j < k$. It remains to deal with equations (1.7) and (1.9) under the assumption that $i + j \geq k$ or $2j \geq k$. As mentioned in Remark 1.3, we consider these equations with the cases $9 \leq i \leq j \leq k$ and $9 \leq j \leq i \leq k$. Let's start with these equations under the condition that $9 \leq i \leq j \leq k$, we respectively obtain that

$$2^{L_j} < 2^{34} + 2^{L_j} \leq 2^{F_i} + 2^{L_j} = F_k^2 \leq F_{2j}^2 \tag{2.4}$$

and

$$2^{L_j} < 2^{34} + 2^{L_j} \leq 2^{F_i} + 2^{L_j} = L_k^2 \leq L_{2j}^2, \tag{2.5}$$

which are not true as shown above. Finally, we obtain the following inequalities from equations (1.7) and (1.9) under the condition that $9 \leq j \leq i \leq k$:

$$2^{F_i} < 2^{76} + 2^{F_i} \leq 2^{F_i} + 2^{L_j} = F_k^2 \leq F_{2i}^2$$

and

$$2^{F_i} < 2^{76} + 2^{F_i} \leq 2^{F_i} + 2^{L_j} = L_k^2 \leq L_{2i}^2,$$

and the falseness of these latter two inequalities was already shown. So, we get contradictions. Hence, we conclude that $i + j < k$ and $2j < k$, and Lemma 2.1 is completely proved. □

Theorem 2.2. *Let $(i, j, k) \in \mathbb{N}^3$ be an arbitrary solution of any of the equations* (1.4)–(1.9)*, then $i, j, k \leq 8$. In particular, equations* (1.4) *to* (1.9) *have five, one, one, two, one and one solutions, respectively.*

Proof. Assume for a contradiction that these equations have solutions only if $i, j, k \geq 9$, then from Lemma 2.1 we have that $i + j < k$ and $2j < k$. Again, as mentioned in Remark 1.3 we fix the condition that $9 \leq i \leq j < k$ in each of the equations (1.4)–(1.6) and (1.8), and in case of equations (1.7) and (1.9) we consider both of the cases $9 \leq i \leq j \leq k$ and $9 \leq j \leq i \leq k$. From equation (1.4) with the use of Benit's formula of Fibonacci numbers in (1.10), we get that

$$(\alpha^2)^k \leq 2 \cdot 5 \cdot 2^{F_j} + 2(\alpha\beta)^k - (\beta^2)^k$$

as $9 \leq i \leq j$. Since $\beta = -1/\alpha$ such that $\alpha = (1+\sqrt{5})/2$ and $\beta = (1-\sqrt{5})/2$, we can rewrite the latter inequality as follows:

$$2^k < (\alpha^2)^k < 12 \cdot 2^{F_j},$$

that leads to

$$k - F_j < \frac{\ln(12)}{\ln(2)} < 3.6. \tag{2.6}$$

From the facts that $i + j < k$, $2j < k$ and $j < F_j$ for all $j \geq 9$, we have that $k - F_j > j$ or $k - F_j > i$. So, inequality (2.6) respectively gives that

$$i \leq 3 \quad \text{or} \quad j \leq 3,$$

which contradicts the assumption of $i, j \geq 9$. Therefore, there are no solutions for equation (1.4) where $i, j, k \geq 9$, and hence $i, j, k \leq 8$. Next, we consider equation (1.5) with the use of Benit's formula of Lucas numbers in (1.10), we obtain that

$$2^k < (\alpha^2)^k \leq 2 \cdot 2^{L_j} - 2(\alpha\beta)^k - (\beta^2)^k < 4 \cdot 2^{L_j},$$

which leads to

$$k - L_j < \frac{\ln(4)}{\ln(2)} = 2.$$

Again, since $i + j < k$, $2j < k$ and $j < L_j$ for all $j \geq 9$, we have that $k - L_j > j$ or $k - L_j > i$. Therefore, we have that

$$i < 2 \quad \text{or} \quad j < 2,$$

and this contradicts the assumption of $i, j \geq 9$. Thus, if equation (1.5) has a solution of the form (i, j, k), then $i, j, k \leq 8$. Following the same approach on equations (1.6) and (1.8), we respectively obtain that

$$2^k < (\alpha^2)^k < 12 \cdot 2^{L_j} \tag{2.7}$$

and

$$2^k < (\alpha^2)^k < 4 \cdot 2^{F_j}. \tag{2.8}$$

Similarly, inequality (2.7) implies that

$$i \leq 3 \quad \text{or} \quad j \leq 3,$$

and inequality (2.8) gives that

$$i < 2 \quad \text{or} \quad j < 2.$$

Again, we get contradictions. Now, we consider equation (1.7) with $9 \leq i \leq j \leq k$, and from the fact that $2^{F_i} < 2^{L_j}$ for all $j \geq 9$ we get that

$$2^k < (\alpha^2)^k < 12 \cdot 2^{L_j}.$$

Again, we have that

$$i \leq 3 \quad \text{or} \quad j \leq 3.$$

It remains to consider equation (1.7) with the case of $9 \leq j \leq i \leq k$. Since $2^{F_i} < 2^{L_i}$ for all $i \geq 9$, we similarly obtain that

$$2^{k - L_i} < 12$$

and hence

$$i \leq 3 \quad \text{or} \quad j \leq 3,$$

since $i + j < k$ and $j \leq i < L_i$ for $i, j \geq 9$ imply that $k - L_i > i$ or j. From both case, we also obtain a contradiction. Therefore, i, j and k must be less than or equal to 8. Finally, we deal with equation (1.9) with the cases $9 \leq i \leq j \leq k$ and $9 \leq j \leq i \leq k$. In fact, by following the exact approach used with equation (1.7) we get that

$$i \leq 2 \quad \text{or} \quad j \leq 2,$$

which clearly leads to a contradiction. Therefore, we again conclude that $i, j, k \leq 8$. Finally, with the help of SageMath software, we easily determine the complete set of solutions to equations (1.4)–(1.9) with $i, j, k \leq 8$ as follows:

Eq.	$\{(i, j, k)\}$
(1.4)	$\{(1, 1, 3), (1, 2, 3), (2, 1, 3), (2, 2, 3), (5, 5, 6)\}$
(1.5)	$\{(2, 2, 3)\}$
(1.6)	$\{(1, 1, 3)\}$
(1.7)	$\{(1, 1, 3), (2, 1, 3)\}$
(1.8)	$\{(4, 4, 3)\}$
(1.9)	$\{(4, 2, 3)\}$

Hence, Theorem 2.2 is completely proved. □

Acknowledgements:

The author would like to express his sincere gratitude to the referees for the careful reading of the manuscript and many useful comments and remarks which improve the quality of the paper.

References

[1] D. Acu, *On a Diophantine equation*, Gen. Math. **15:4** (2007), 145–148. 1
[2] Z. F. Cao, *A note on the Diophantine equation* $a^x + b^y = c^z$, Chinese Sci. Bull. **II, ibid. 33** (1988), 1519–1521. 1
[3] Z. F. Cao, *A note on the Diophantine equation* $a^x + b^y = c^z$, Acta Arith. **91:1** (1999), 85–93. 1
[4] T. Hadano, *On the Diophantine equation* $a^x = b^y + c^z$, Math. J. Okayama Univ. **19:1** (1976), 25–29. 1
[5] H. R. Hashim, *Solutions of the Diophantine equation* $7X^2 + Y^7 = Z^2$ *from recurrence sequences*, Commun. Math. **28:1** (2020), 55–66. 1
[6] H. R. Hashim and Sz. Tengely, *Representations of reciprocals of Lucas sequences*, Miskolc Math. Notes **19:2** (2018), 865–872. 1
[7] T. Koshy, *Polynomial extensions of two Gibonacci delights and their graph-theoretic confirmations*, Math. Sci. **43:2** (2018), 96–108. 1
[8] A. Makowski, *On the Diophantine equation* $2^x + 11^y = 5^z$, Nordisk Mat. Tidskr. **7:1** (1959), 81-96. 1
[9] T. Nagell, *Sur une classe d'équations exponentielles*, Ark. Mat. **3** (1958), 569–582. 1
[10] A. Pethő, *Diophantine properties of linear recursive sequences. II*, Acta Math. Acad. Paedagog. Nyházi. (N.S.) **17** (2001), 81–96. 1
[11] J. F. T. Rabago, *On the Diophantine equation* $2^x + 17^y = z^2$, J. Indones. Math. Soc. **22:2** (2016), 177–181. 1
[12] P. Ribenboim, *My numbers, my friends. Popular lectures on number theory*, New York, NY: Springer (2000), ix + 375. 1
[13] J. Sándor, *Geometric theorems, Diophantine equations, and arithmetic functions*, Rehoboth, NM: American Research Press (2002). 1
[14] A. Srinivasan, *The Markoff-Fibonacci numbers*, Fibonacci Q. **58:5** (2020), 222–228. 1
[15] W. A. Stein and others, *Sage Mathematics Software (Version 9.0)*, the Sage Development Team (2020), http://www.sagemath.org 1.3
[16] Q. Sun and X. Zhou, *On the Diophantine equation* $a^x + b^y = c^z$, J. Sichuan Univ., Nat. Sci. Ed. **4** (1984), 36–38. 1
[17] A. Suvarnamani, *Solutions of the Diophantine equation* $2^x + p^y = z^2$, Int. J. Math. Sci. Appl. **1:3** (2011), 1415–1419. 1
[18] S. Uchiyama, *On the Diophantine equation* $2^x = 3^y + 13^z$, Math. J. Okayama Univ. **19** (2011), 31–38. 1
[19] X. Yang, *On the Diophantine equation* $a^x + b^y = c^z$, J. Sichuan Univ., Nat. Sci. Ed. **4** (1985), 151–158. 1