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Solving Split Equality Fixed Point of Nonexpansive Semigroup and split equality minimization Problems in real Hilbert Space

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Abstract

In this article, we study the split equality problem involving nonexpansive semigroup and convex minimization problem. Using a Halpern iterative algorithm, we establish a strong convergence result for approximating a common solution of the aforementioned problems. The iterative algorithm introduced in this paper is designed in such a way that it does not require the knowledge of the operator norm. We display a numerical example to show the relevance of our result. Our result complements and extends some related results in literature.

Keywords: Split equality minimization problem; semigroup nonexpansive; iterative scheme; Fixed point problem.
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1. Introduction

The Minimization Problem (MP) is one of the most important problems in optimization theory and non-linear analysis. The MP is defined as follows: find $x \in H$ such that

$$\psi(x) := \min_{y\in H}\psi(y), \tag{1.1}$$

where $\psi : H \to (-\infty, \infty]$ is a proper and convex function. We denote by $argmin_{y\in H}\psi(y)$ the set of all minimizers of ψ on H.

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Numerous problems in signal processing and imaging, statistical learning and data mining or computer vision can be formulated as optimization problem that consists of a sum of convex functions which may not be necessarily differentiable, possibly composed with linear operators and that in turn can be transformed to Split Minimization Problem (SMP), see for example [1, 6].
Let H_1, H_2 and H_3 be real Hilbert spaces and $A : H_1 \to H_3, B : H_2 \to H_3$ be bounded linear operators. Let C and Q be nonempty, closed and convex subsets of H_1 and H_2 respectively, the Split Equality Problem (SEP) introduced by Moudafi [13] is to find

$$x^* \in C, y^* \in Q \text{ such that } Ax^* = By^*. \tag{1.2}$$

A point $x \in C$ is called a fixed point of a single-valued mapping T if $x = Tx$. We denote by $F(T)$, the set of all fixed points of T.
In [15], Moudafi introduced the following Split Equality Fixed Point Problem (SEFPP): Let $T : H_1 \to H_1$ and $S : H_2 \to H_2$ be nonlinear operators such that $F(T) \neq \emptyset$ and $F(S) \neq \emptyset$. If $C = F(T)$ and $Q = F(S)$ in (1.2), then the SEFPP is to find:

$$x^* \in F(T) \text{ and } y^* \in F(S) \text{ such that } Ax^* = By^*. \tag{1.3}$$

Since the inception of SEFPP (1.3), many authors working in this direction have used SEFPP (1.3) to solve different optimization problems (see [2, 3, 4, 5, 7, 10, 11, 14, 16, 19, 18, 22] and the references therein).
Let C and Q be nonempty closed and convex subset of real Hilbert spaces H_1 and H_2 respectively, $\psi : H_1 \to \mathbb{R} \cup \{+\infty\}$ and $\varphi : H_2 \to \mathbb{R} \cup \{+\infty\}$ be two proper and lower semi-continuous convex functions and $A : H_1 \to H_2$ a bounded linear operator. The SMP is to find

$$\begin{aligned} &x^* \in C \text{ such that } x^* = \text{argmin}_{x \in C} \psi(x); \text{ and} \\ &y^* = Ax^* \in Q \text{ solves } y^* = \text{argmin}_{z \in Q} \varphi(z). \end{aligned} \tag{1.4}$$

It is well-known that

$$x \in \text{argmin}\psi \Longleftrightarrow J_\mu^\psi(x) := \text{argmin}_u \{\psi(u) + \frac{1}{2\mu} ||u - x||^2\}.$$

The fixed point set of proximity mapping is precisely the set of minimizers of ψ. Based on (1.4), the Split Equality Minimization Problem (SEMP) is to find

$$x^* \in \text{argmin}\psi, y^* \in \text{argmin } \varphi \text{ such that } Ax^* = By^*, \tag{1.5}$$

hence (x^*, y^*) solves

$$\min_{x,y} \{\psi(x) + \varphi(y) + \frac{1}{2} ||Ax - By||^2\},$$

an optimization problem with weak coupling in the objective function as well as

$$\min_{x,y} \{\psi(x) + \varphi(y), \; Ax = By\}.$$

The metric projection P_C is a map defined on H onto C which assigns to each $x \in H$, the unique point in C, denoted by $P_C x$ such that

$$\|x - P_C x\| = \inf\{\|x - y\| : y \in C\}.$$

It is well known that $P_C x$ is characterized by the inequality $\langle x - P_C x, z - P_C x \rangle \leq 0, \; \forall \; z \in C$ and P_C is a firmly nonexpansive mapping.
Motivated by the works of authors mentioned above, we introduce an iterative algorithm that does not require the knowledge of operator norm to approximate a common solution of split equality minimization problem and split equality fixed point problem of nonexpansive semigroup in real Hilbert space. We also prove a strong convergence result for approximating a common solution of the aforementioned problems.

2. Preliminaries

We state some known and useful results which will be needed in the proof of our main theorem. In the sequel, we denote strong and weak convergence by "$\rightarrow$" and "$\rightharpoonup$", respectively.
One-parameter family mapping $S = \{T(s) : 0 \leq s < \infty\}$ from H into itself is said to be a nonexpansive semigroup (see [8]), if it satisfies the following conditions:
(i) $T(0)x = x$, for all $x \in H$;
(ii) $T(s + t) = T(s)T(t)$, for all $s, t \geq 0$;
(iii) For each $x \in H$, the mapping $T(.)x$ is continuous.
(iv) $||T(s)x - T(s)y|| \leq ||x - y||$, for all $x, y \in H$ and $s \geq 0$.

Lemma 2.1. *[9] Let H be a real Hilbert space, then*

$$2\langle x, y\rangle = ||x||^2 + ||y||^2 - ||x - y||^2 = ||x + y||^2 - ||x||^2 - ||y||^2, \ \forall \ x, y \in H.$$

Lemma 2.2. *[20] Let C be a nonempty, closed and convex subset of a real Hilbert space H and $S : C \rightarrow C$ be a nonexpansive mapping. Then $I - T$ is demiclosed at 0 (i.e., if $\{x_n\}$ converges weakly to $x \in C$ and $\{x_n - Tx_n\}$ converges strongly to 0, then $x = Tx$).*

Lemma 2.3. *[9] Let H be a Hilbert space, then $\forall \ x, y \in H$ and $\alpha \in \mathbb{R}$, we have*

$$(i) \ ||\alpha x + (1 - \alpha)y||^2 = \alpha||x||^2 + (1 - \alpha)||y||^2 - \alpha(1 - \alpha)||x - y||^2.$$
$$(ii) \ ||x + y||^2 \leq ||x||^2 + 2\langle y, x + y\rangle.$$

Lemma 2.4. *[8] Let C be a nonempty closed and convex subset of a Hilbert space and $\{T(s)\}_{s\geq 0}$ be a nonexpansive semigroup on H. Then, for every $h \geq 0$*

$$\limsup_{t\rightarrow\infty} x \in C \left|\left|\frac{1}{t}\int_0^t T(s)xds - T(h)\left(\frac{1}{t}\int_0^t T(s)xds\right)\right|\right| = 0.$$

Lemma 2.5. *[12] Let H be a real Hilbert space and $f : H \rightarrow (-\infty, \infty]$ be a proper convex and lower semi-continuous function. Then, for all $x, y \in H$ and $\lambda > 0$, we have*

$$\frac{1}{2\lambda}||J_\lambda x - y||^2 - \frac{1}{2\lambda}||x - y||^2 + \frac{1}{2\lambda}||x - J_\lambda x||^2 + f(J_\lambda x) \leq f(y).$$

Lemma 2.6. *[16] Let H be a real Hilbert space and $f : H \rightarrow (-\infty, \infty]$ be a proper convex and lower semi-continuous function. Then, for all $0 < \lambda \leq \mu$ and $x \in \mathbb{N}$, we have*

$$||J_\lambda x - x|| \leq |J_\mu x - x||.$$

Lemma 2.7. *[21] Assume $\{a_n\}$ is a sequence of nonnegative real sequence such that*

$$a_{n+1} \leq (1 - \sigma_n)a_n + \sigma_n\delta_n, \ n > 0,$$

where $\{\sigma_n\}$ is a sequence in $(0, 1)$ and $\{\delta_n\}$ is a real sequence such that
(i) $\sum\limits_{n=1}^{\infty} \sigma_n = \infty$,
(ii) $\limsup\limits_{n\rightarrow\infty} \delta_n \leq 0$ *or* $\sum\limits_{n=1}^{\infty} |\sigma_n\delta_n| < \infty$.
Then $\lim\limits_{n\rightarrow\infty} a_n = 0$.

3. Main Result

In this section, we state and prove our main result. We denote by J_{μ}^{f}, the resolvent of MP.

Lemma 3.1. *Let H_1, H_2 and H_3 be real Hilbert spaces, $A : H_1 \to H_3$ and $B : H_2 \to H_3$ be bounded linear operators. Let $\psi : H_1 \to (-\infty, +\infty], \varphi : H_2 \to (-\infty, +\infty]$ be two proper, convex and lower semi-continuous functions and $\{T(s) : 0 \leq s < \infty\}, \{R(m) : 0 \leq m < \infty\}$ be two-parameters nonexpansive semigroups on H_1 and H_2 respectively. Suppose $\Gamma := \{p \in F(T(s)) \cap argmin_{y\in H_1}\psi(y), q \in F(R(m)) \cap argmin_{y\in H_2}\varphi(y)$ and $Ap = Bq\} \neq \emptyset$ and the step size sequence $\{\gamma_n\}$ is chosen in such a way that for some $\epsilon > 0$*

$$\gamma_n \in \left(\epsilon, \frac{2||Aw_n - Bz_n||^2}{||A^*(Aw_n - Bz_n)||^2 + ||B^*(Aw_n - Bz_n)||^2} - \epsilon\right),\ n \in \Omega.$$

Otherwise $\gamma_n = \gamma$(γ being any any nonnegative value), where the set of indexes $\Omega = \{n : Aw_n - Bz_n \neq 0\}$. Let $u, x_1 \in H_1$ and $v, y_1 \in H_2$ be arbitrary and the sequence $(\{x_n\}, \{y_n\})$ be generated iteratively by

$$\begin{cases} w_n = (1-\alpha_n)x_n + \alpha_n u; \\ z_n = (1-\alpha_n)y_n + \alpha_n v; \\ u_n = J_{\rho_n}^{\psi}(w_n - \gamma_n A^*(Aw_n - Bz_n)); \\ v_n = J_{\mu_n}^{\varphi}(z_n + \gamma_n B^*(Aw_n - Bz_n)); \\ x_{n+1} = (1-\beta_n)u_n + \beta_n \frac{1}{t_n}\int_0^{t_n} T(s)u_n ds; \\ y_{n+1} = (1-\beta_n)v_n + \beta_n \frac{1}{r_n}\int_0^{r_n} R(m)v_n dm; \end{cases} \tag{3.1}$$

where $\{\alpha_n\}, \{\beta_n\}$ are sequences in $(0,1)$, A^* and B^* are adjoints of A and B respectively. Let $0 < \rho \leq \rho_n$, $0 < \mu \leq \mu_n$ and $\{t_n\}, \{r_n\}$ be sequences in $[0, \infty)$, then $\{x_n\}$ and $\{y_n\}$ are bounded.

Proof. Let $(p, q) \in \Gamma$, $a_n = w_n - \gamma_n A^*(Aw_n - Bz_n)$ and $b_n = z_n + \gamma_n B^*(Aw_n - Bz_n)$ then from (3.1) and Lemma 2.1, we have that

$$\begin{aligned} ||u_n - p||^2 &= ||J_{\rho_n}^{\psi} a_n - p||^2 \\ &\leq ||a_n - p||^2 \\ &= ||w_n - p||^2 + \gamma_n^2||A^*(Aw_n - Bz_n)||^2 - 2\gamma_n\langle w_n - p, A^*(Aw_n - Bz_n)\rangle \\ &= ||w_n - p||^2 + \gamma_n^2||A^*(Aw_n - Bz_n)||^2 - 2\gamma_n\langle Aw_n - Ap, Aw_n - Bz_n)\rangle \\ &= ||w_n - p||^2 + \gamma_n^2||A^*(Aw_n - Bz_n)||^2 - \gamma_n||Aw_n - Ap||^2 \\ &- \gamma_n||Aw_n - Bz_n||^2 + \gamma_n||Bz_n - Ap||^2. \end{aligned} \tag{3.2}$$

By similar steps as in (3.2), we have

$$\begin{aligned} ||v_n - q||^2 &\leq \|b_n - q\|^2 \\ &= ||z_n - q||^2 + \gamma_n^2||B^*(Aw_n - Bz_n)||^2 - \gamma_n||Bz_n - Bq||^2 \\ &- \gamma_n||Aw_n - Bz_n||^2 + \gamma_n||Aw_n - Bq||^2. \end{aligned} \tag{3.3}$$

Adding (3.2) and (3.3), using the fact that $Ap = Bq$ and noting the assumption on γ_n, we obtain

$$\begin{aligned} ||u_n - p||^2 + ||v_n - q||^2 &\leq \|a_n - p\|^2 + \|b_n - q\|^2 \\ &\leq ||w_n - p||^2 + ||z_n - q||^2 - \gamma_n[2||Aw_n - Bz_n||^2 \\ &- \gamma_n(||A^*(Aw_n - Bz_n)||^2 + ||B^*(Aw_n - Bz_n||^2)] \\ &\leq ||w_n - p||^2 + ||z_n - q||^2. \end{aligned} \tag{3.4}$$

From (3.1), we have that

$$\begin{aligned}||w_n - p||^2 &= ||(1-\alpha_n)x_n + \alpha_n u - p||^2\\ &= ||(1-\alpha_n)(x_n - p) + \alpha_n(u-p)||^2\\ &\leq (1-\alpha_n)||x_n - p||^2 + \alpha_n||u-p||^2. \end{aligned}\tag{3.5}$$

Using the same approach in (3.5), we have that

$$||z_n - q||^2 \leq (1-\alpha_n)||y_n - q||^2 + \alpha_n||v-q||^2. \tag{3.6}$$

Adding (3.5) and (3.6), we have

$$||w_n - p||^2 + ||z_n - q||^2 \leq (1-\alpha_n)[||x_n - p||^2 + ||y_n - q||^2] + \alpha_n[||u-p||^2 + ||v-q||^2]. \tag{3.7}$$

From (3.1) and Lemma 2.3, we have that

$$\begin{aligned}&||x_{n+1} - p||^2 + ||y_{n+1} - q||^2\\ &= ||(1-\beta_n)u_n + \beta_n\frac{1}{t_n}\int_0^{t_n} T(s)u_n ds - p||^2 + \|(1-\beta_n)v_n + \beta_n\frac{1}{r_n}\int_0^{r_n} R(m)v_n dm - q\|^2\\ &= ||(1-\beta_n)(u_n - p) + \beta_n(\frac{1}{t_n}\int_0^{t_n} T(s)u_n ds - \frac{1}{t_n}\int_0^{t_n} T(s)pds)||^2\\ &+ ||(1-\beta_n)(v_n - q) + \beta_n(\frac{1}{r_n}\int_0^{r_n} R(m)v_n dm - \frac{1}{r_n}\int_0^{r_n} R(m)qdm)||^2\\ &\leq (1-\beta_n)||u_n - p||^2 + \beta_n\left|\left|\frac{1}{t_n}\int_0^{t_n} T(s)u_n ds - \frac{1}{t_n}\int_0^{t_n} T(s)pds\right|\right|^2\\ &- \beta_n(1-\beta_n)\left|\left|\frac{1}{t_n}\int_0^{t_n} T(s)u_n ds - u_n\right|\right|^2\\ &+ (1-\beta_n)||v_n - q||^2 + \beta_n\left|\left|\frac{1}{r_n}\int_0^{r_n} R(m)v_n dm - \frac{1}{r_n}\int_0^{r_n} R(m)qdm\right|\right|^2\\ &- \beta_n(1-\beta_n)\left|\left|\frac{1}{r_n}\int_0^{r_n} R(m)v_n dm - v_n\right|\right|^2\\ &\leq (1-\beta_n)||u_n - p||^2 + \beta_n||u_n - p||^2 - \beta_n(1-\beta_n)\left|\left|\frac{1}{t_n}\int_0^{t_n} T(s)u_n ds - u_n\right|\right|^2\\ &+ (1-\beta_n)||v_n - q||^2 + \beta_n||v_n - q||^2 - \beta_n(1-\beta_n)\left|\left|\frac{1}{r_n}\int_0^{r_n} R(m)v_n dm - v_n\right|\right|^2\\ &= ||u_n - p||^2 + \|v_n - q\|^2 - \beta_n(1-\beta_n)\Bigg[\left|\left|\frac{1}{t_n}\int_0^{t_n} T(s)u_n ds - u_n\right|\right|^2\\ &+ \left|\left|\frac{1}{r_n}\int_0^{r_n} R(m)v_n dm - v_n\right|\right|^2\Bigg]\\ &\leq ||u_n - p||^2 + \|v_n - q\|^2. \end{aligned}\tag{3.8}$$

Using (3.1), (3.7) and (3.8), we have that

$$\begin{aligned}||x_{n+1} - p||^2 + ||y_{n+1} - q||^2 &\leq ||u_n - q||^2 + ||v_n - q||^2\\ &\leq ||w_n - p||^2 + ||z_n - q||^2\\ &\leq (1-\alpha_n)[||x_n - p||^2 + ||y_n - q||^2] + \alpha_n[||u-p||^2 + ||v-q||^2]\\ &\leq \max\{||x_n - p||^2 + ||y_n - q||^2, ||u-p||^2 + ||v-q||^2\}\\ &\vdots\\ &\leq \max\{||x_1 - p||^2 + ||y_1 - q||^2, ||u-p||^2 + ||v-q||^2\}. \end{aligned}$$

Therefore, $\{||x_{n+1}-p||^2\}+\{||y_{n+1}-q||^2\}$ is bounded. Hence, $\{x_n\}$ and $\{y_n\}$ are bounded. Consequently, $\{u_n\},\{v_n\},\{w_n\},\{z_n\},\{\frac{1}{t_n}\int_0^{t_n}T(s)u_nds\}$ and $\{\frac{1}{r_n}R(m)v_ndm\}$ are also bounded.

Theorem 3.2. *Suppose that Lemma 3.1 holds and let $0<\rho\le\rho_n$, $0<\mu\le\mu_n$ and $\{s_n\},\{r_n\}$ be sequences in $[0,\infty)$ with conditions:*
(i) $\lim_{n\to\infty}\alpha_n=0$ *and* $\sum_{n=1}^{\infty}\alpha_n=\infty$;
(ii) $0<\liminf_{n\to\infty}\beta_n<\limsup_{n\to\infty}\beta_n<1.$
Then $(\{x_n\},\{y_n\})$ converges strongly to $(\overline{x},\overline{y})\in\Gamma$.

From (3.1), (3.2) and (3.7), we have that

$$\begin{aligned}||x_{n+1}-p||^2+||y_{n+1}-q||^2 &\le ||w_n-p||^2+||z_n-q||^2-\gamma_n[2||Aw_n-Bz_n||^2\\&\quad-\gamma_n(||A^*(Aw_n-Bz_n)+||B^*(Aw_n-Bz_n)||^2)]\\&\le(1-\alpha_n)[||x_n-p||^2+||y_n-q||^2]+\alpha_n[||u-p||^2+||v-q||^2]\\&\quad-\gamma_n[2||Aw_n-Bz_n||^2-\gamma_n(||A^*(Aw_n-Bz_n)+||B^*(Aw_n-Bz_n)||^2)].\end{aligned}\tag{3.9}$$

Case 1: Assume that $\{||x_n-p||^2+||y_n-q||^2\}$ is monotone decreasing, then $\{||x_n-p||^2+||y_n-q||^2\}$ is convergent, thus $\lim_{n\to\infty}[(||x_{n+1}-p||^2+||y_{n+1}-q||^2)-(||x_n-p||^2+||y_n-q||^2)]=0.$
From (3.9), we have that

$$\begin{aligned}&\gamma_n^2(||A^*(Aw_n-Bz_n)+||B^*(Aw_n-Bz_n)||^2)\\&\le(1-\alpha_n)[||x_n-p||^2+||y_n-q||^2][||x_{n+1}-p||^2+||y_{n+1}-q||^2]+\alpha_n[||u-p||^2+||v-q||^2].\end{aligned}\tag{3.10}$$

From condition (i) of Theorem 3.2 and the condition

$$\gamma_n\in\left(\epsilon,\frac{2||Aw_n-Bz_n||^2}{||A^*(Aw_n-Bz_n)||^2+||B^*(Aw_n-Bz_n)||^2}-\epsilon\right),\ n\in\Omega.$$

We conclude that

$$(||A^*(Aw_n-Bz_n)||^2+||B^*(Aw_n-Bz_n)||^2)\to0,\ \text{as } n\to\infty.$$

Since $Aw_n-Bz_n=0$, if $n\in\Omega$, therefore we have

$$\lim_{n\to\infty}||A^*(Aw_n-Bz_n)||^2=\lim_{n\to\infty}||B^*(Aw_n-Bz_n)||^2=0.\tag{3.11}$$

From (3.1), we have

$$\begin{aligned}||w_n-x_n||&=||(1-\alpha_n)(x_n-x_n)+\alpha_n(u-x_n)||\\&\le\alpha_n||u-x_n||\end{aligned}$$

From condition (ii) of Theorem 3.2, we have that

$$\lim_{n\to\infty}||w_n-x_n||=0.\tag{3.12}$$

Also,

$$\begin{aligned}||z_n-y_n||&=||(1-\alpha_n)(y_n-y_n)+\alpha_n(v-y_n)||\\&\le\alpha_n||v-y_n||\end{aligned}$$

From condition (i) of (3.2), we have that

$$\lim_{n\to\infty}||z_n-y_n||=0.\tag{3.13}$$

From (3.1), (3.4) and (3.7), we have that

$$
\begin{aligned}
||x_{n+1}-p||^2+\|y_{n+1}-q\|^2 &\le ||u_n-p||^2+\|v_n-q\|^2-\beta_n(1-\beta_n)\Big[||\frac{1}{t_n}\int_0^{t_n}T(s)u_nds-u_n||^2\\
&\quad+||\frac{1}{r_n}\int_0^{r_n}R(m)v_ndm-v_n||^2\Big]\\
&\le \|w_n-p\|^2+\|z_n-q\|^2-\beta_n(1-\beta_n)\Big[||\frac{1}{t_n}\int_0^{t_n}T(s)u_nds-u_n||^2\\
&\quad+||\frac{1}{r_n}\int_0^{r_n}R(m)v_ndm-v_n||^2\Big]\\
&\le (1-\alpha_n)[||x_n-p||^2+||y_n-q||^2]+\alpha_n[||u-p||^2+||v-q||^2]\\
&\quad-\beta_n(1-\beta_n)\Big[||\frac{1}{t_n}\int_0^{t_n}T(s)u_nds-u_n||^2+||\frac{1}{r_n}\int_0^{r_n}R(m)v_ndm-v_n||^2\Big],
\end{aligned}
\tag{3.14}
$$

which implies that

$$
\begin{aligned}
&\beta_n(1-\beta_n)\Big[||\frac{1}{t_n}\int_0^{t_n}T(s)u_nds-u_n||^2+||\frac{1}{r_n}\int_0^{r_n}R(m)v_ndm-v_n||^2\Big]\\
&\le (1-\alpha_n)[||x_n-p||^2+||y_n-q||^2]-[||x_{n+1}-p||^2+||y_{n+1}-q||^2]+\alpha_n[||u-p||^2+||v-q||^2].
\end{aligned}
$$

From condition (i) of Theorem 3.2, we have that

$$
||\frac{1}{t_n}\int_0^{t_n}T(s)u_nds-u_n||^2+||\frac{1}{r_n}\int_0^{r_n}R(m)v_ndm-v_n||^2\to 0.
$$

Hence,

$$
\lim_{n\to\infty}||\frac{1}{t_n}\int_0^{t_n}T(s)u_nds-u_n||=||\frac{1}{r_n}\int_0^{r_n}R(m)v_ndm-v_n||=0. \tag{3.15}
$$

From Lemma 2.5, we have that

$$
\frac{1}{2\rho_n}||u_n-p||^2+\frac{1}{2\rho_n}||a_n-p||^2+\frac{1}{2\rho_n}||a_n-u_n||^2\le\psi(y)-\psi(u_n).
$$

Since $\psi(p)\le\psi(u_n)$ for all $n\ge 1$, we obtain

$$
||u_n-p||^2\le||a_n-p||^2-||a_n-u_n||^2. \tag{3.16}
$$

Similarly, from (2.5), we have that

$$
\frac{1}{2\mu_n}||v_n-q||^2+\frac{1}{2\mu_n}||b_n-q||^2+\frac{1}{2\mu_n}||b_n-v_n||^2\le g(y)-g(v_n).
$$

Since $g(q)\le g(v_n)$ for all $n\ge 1$, we obtain

$$
||v_n-q||^2\le||b_n-q||^2-||b_n-v_n||^2. \tag{3.17}
$$

By adding (3.16) and (3.17), we get

$$
\|u_n-p\|^2+\|v_n-q\|^2\le\|a_n-p\|^2+\|b_n-q\|^2-\big[\|a_n-u_n\|^2+\|b_n-v_n\|^2\big]. \tag{3.18}
$$

On substituting (3.18) into (3.8), and applying (3.4) and (3.7), we obtain

$$
\begin{aligned}
||x_{n+1}-p||^2+\|y_{n+1}-q\|^2 &\leq ||u_n-p||^2+\|v_n-q\|^2-\big[\|a_n-u_n\|^2+\|b_n-v_n\|^2\big]\\
&\leq \|a_n-p\|^2+\|b_n-q\|^2-\big[\|a_n-u_n\|^2+\|b_n-v_n\|^2\big]\\
&\leq \|w_n-p\|^2+\|z_n-q\|^2-\big[\|a_n-u_n\|^2+\|b_n-v_n\|^2\big]\\
&\leq (1-\alpha_n)[||x_n-p||^2+||y_n-q||^2]+\alpha_n[||u-p||^2+||v-q||^2]\\
&-\big[\|a_n-u_n\|^2+\|b_n-v_n\|^2\big], \qquad (3.19)
\end{aligned}
$$

which implies that

$$
\begin{aligned}
||u_n-a_n||^2+||v_n-b_n||^2 \leq (1-\alpha_n)[||x_n-p||^2+||y_n-q||^2]-[\|x_{n+1}-p\|^2+\|y_{n+1}-q\|^2]\\
+\alpha_n[||u-p||^2+||v-q||^2].
\end{aligned}
$$

Hence, we obtain that

$$\lim_{n\to\infty}||u_n-a_n||=\lim_{n\to\infty}||v_n-b_n||=0. \qquad (3.20)$$

Using the definition of a_n, b_n and applying (3.11), we have that

$$\lim_{n\to\infty}||a_n-w_n||^2=\lim_{n\to\infty}\gamma_n^2||A^*(Aw_n-Bz_n)||^2=0. \qquad (3.21)$$

Similarly,

$$\lim_{n\to\infty}||b_n-z_n||^2=\lim_{n\to\infty}\gamma_n^2||B^*(Aw_n-Bz_n)||^2=0. \qquad (3.22)$$

From (3.21) and (3.22), we have that

$$\lim_{n\to\infty}||u_n-w_n||=0=\lim_{n\to\infty}\|v_n-z_n\|. \qquad (3.23)$$

Also, from (3.12), (3.13) and (3.23), we have

$$\lim_{n\to\infty}\|u_n-x_n\|=||v_n-y_n||=0. \qquad (3.24)$$

From (3.1), we obtain

$$
\begin{aligned}
||x_{n+1}-u_n|| &= ||(1-\beta_n)u_n+\beta_n\frac{1}{t_n}\int_0^{t_n}T(s)u_n ds-u_n||\\
&\leq (1-\beta_n)||u_n-u_n||+\beta_n\Big|\Big|\frac{1}{t_n}\int_0^{t_n}T(s)u_n ds-u_n\Big|\Big|.
\end{aligned}
$$

Thus, from (3.15), we have that

$$\lim_{n\to\infty}||x_{n+1}-u_n||=0. \qquad (3.25)$$

Similarly, from (3.1), we have that

$$
\begin{aligned}
||y_{n+1}-v_n|| &= ||(1-\beta_n)v_n+\beta_n\frac{1}{r_n}\int_0^{r_n}R(m)v_n dm-v_n||\\
&\leq (1-\beta_n)||v_n-v_n||+\beta_n\Big|\Big|\frac{1}{r_n}\int_0^{r_n}R(m)v_n dm-v_n\Big|\Big|.
\end{aligned}
$$

Thus, from (3.15), we have obtain

$$\lim_{n\to\infty} ||y_{n+1} - v_n|| = 0. \tag{3.26}$$

From (3.1), we have that

$$||x_{n+1} - x_n|| \leq ||x_{n+1} - u_n|| + ||u_n - x_n||.$$

Thus, from (3.20) and (3.25), we obtain

$$\lim_{n\to\infty} ||x_{n+1} - x_n|| = 0. \tag{3.27}$$

Also, from (3.1), we obtain

$$||y_{n+1} - y_n|| \leq ||y_{n+1} - v_n|| + ||v_n - y_n||$$

Thus, from (3.20) and (3.26), we obtain

$$||y_{n+1} - y_n|| = 0. \tag{3.28}$$

Note that

$$\begin{aligned}||u_n - T(u)u_n|| \leq & \left|\left| u_n - \frac{1}{t_n}\int_0^{t_n} T(s)u_n ds \right|\right| + \left|\left| \frac{1}{t_n}\int_0^{t_n} T(s)u_n ds - T(u)\frac{1}{t_n}\int_0^{t_n} T(s)u_n ds \right|\right| \\ & + \left|\left| T(u)\frac{1}{t_n}\int_0^{t_n} T(s)u_n ds - T(u)u_n \right|\right| \\ & \leq 2\left|\left| u_n - \frac{1}{t_n}\int_0^{t_n} T(s)u_n ds \right|\right| + \left|\left| \frac{1}{t_n}\int_0^{t_n} T(s)u_n ds - T(u)\frac{1}{t_n}\int_0^{t_n} T(s)u_n ds \right|\right|.\end{aligned}$$

It follows from (3.15) and Lemma 2.4 that

$$\lim_{n\to\infty} ||u_n - T(u)u_n|| = 0. \tag{3.29}$$

Similarly, using the same approach as in (3.29), we have that

$$\lim_{n\to\infty} ||v_n - T(v)v_n|| = 0. \tag{3.30}$$

Since $\rho_n > \rho > 0$, we have from Lemma 2.6 and (3.20) that

$$||J_\rho a_n - a_n|| \leq ||J_{\rho_n} a_n - a_n||.$$

Hence,

$$\lim_{n\to\infty} ||J_\rho a_n - a_n|| = 0. \tag{3.31}$$

Similarly, from Lemma 2.6 and (3.20), we have that

$$||J_\mu b_n - b_n|| \leq ||J_{\mu_n} b_n - b_n||.$$

Thus, we obtain that

$$\lim_{n\to\infty} ||J_\mu b_n - b_n|| = 0. \tag{3.32}$$

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ which converges weakly to $\overline{x}$. It follows from (3.12) and (3.20) that the subsequences $\{u_{n_j}\}$ and $\{w_{n_j}\}$ of $\{u_n\}$ and $\{w_n\}$ converges weakly to $\overline{x}$.

Similarly, since $\{y_n\}$ is bounded, there exists a subsequence $\{y_{n_j}\}$ of $\{y_n\}$ which converges weakly to $\overline{y}$. From (3.13) and (3.20), we have that subsequences $\{v_{n_j}\}$, $\{z_{n_j}\}$ of $\{v_n\}$ and $\{z_n\}$ also converges weakly to $\overline{y}$. Hence, from the nonexpansiveness of J_ρ, it follows from the demiclosedness principle (Lemma 2.2) and (3.31) that $\overline{x} \in F(J^{\psi}_{\rho})$. Following the same approach and using (3.32), we have that $\overline{y} \in F(J^{\varphi}_{\mu})$. Using (3.29), (3.30) and Lemma 2.2, we have that $\overline{x} \in F(T(s))$ $\overline{y} \in F(R(m))$.
Next, we show that $A\overline{x} = B\overline{y}$. Since A and B are bounded linear operators, we have $Aw_n \rightharpoonup A\overline{x}$ and $Bz_n \rightharpoonup B\overline{y}$.
Using the condition on $\{\gamma_n\}$ and (3.9), we have that

$$\lim_{n\to\infty} ||Aw_n - Bz_n||^2 = 0. \tag{3.33}$$

By weakly semi continuity of the norm, we have

$$||A\overline{x} - B\overline{y}|| \leq \liminf_{n\to\infty} ||Aw_n - Bz_n|| = 0. \tag{3.34}$$

Thus,

$$A\overline{x} = B\overline{y}. \tag{3.35}$$

Now, since $\{x_{n_j}\}$ converges weakly to $\overline{x}$, we obtain by the property of P_Γ that Next, we show that $(\{x_n\}, \{y_n\})$ converges strongly to $(\overline{x}, \overline{y})$.
Now, since $(\{x_{n_j}\}, \{y_{n_j}\})$ converges weakly to $(\overline{x}, \overline{y})$, we obtain by the property of P_Γ that

$$\begin{aligned}\limsup_{n\to\infty}\langle u-p, x_n - p\rangle + \limsup_{n\to\infty}\langle v-q, y_n-q\rangle &= \lim_{j\to\infty}\langle u-p, x_{n_j}-p\rangle + \lim_{j\to\infty}\langle v-q, y_{n_j}-q\rangle \\ &= \langle u-p, \overline{x}-p\rangle + \langle v-q, \overline{y}-q\rangle \\ &\leq 0. \end{aligned}\tag{3.36}$$

From (3.4), we have that

$$\begin{aligned}||x_{n+1}-\overline{x}||^2 + ||y_{n+1}-\overline{y}||^2 &\leq ||w_n - \overline{x}||^2 + ||z_n - \overline{y}|^2 \\ &= (1-\alpha_n)^2||x_n-\overline{x}||^2 + \alpha_n^2||u-\overline{x}||^2 + 2(1-\alpha_n)\alpha_n\langle x_n - \overline{x}, u-\overline{x}\rangle \\ &+ (1-\alpha_n)^2||y_n-\overline{y}||^2 + \alpha_n^2||v-\overline{y}||^2 + 2(1-\alpha_n)\alpha_n\langle y_n-\overline{y}, v-\overline{y}\rangle \\ &\leq (1-\alpha_n)[||x_n-\overline{x}||^2 + ||y_n-\overline{y}||^2] + \alpha_n[\alpha_n||u-\overline{x}||^2 + 2(1-\alpha_n)\langle x_n-\overline{x}, u-\overline{x}\rangle \\ &+ \alpha_n||v-\overline{y}||^2 + 2(1-\alpha_n)\langle y_n - \overline{y}, v-\overline{y}\rangle]. \end{aligned}\tag{3.37}$$

Applying Lemma 2.7, (3.36) and condition (i) of Theorem (3.2), we have that $(\{x_n\}, \{y_n\})$ converges strongly to $(\overline{x}, \overline{y})$.
Case 2: Assume that $\{||x_n-p||^2+||y_n-q||^2\}$ is not monotone decreasing. Suppose $\Upsilon_n := ||x_n-p||^2+||y_n-q||^2$ and let $\tau : \mathbb{N} \to \mathbb{N}$ be a mapping defined for all $n \geq n_0$ (for some large n_0) by

$$\tau(n) := \max\{k \in \mathbb{N} : k \leq n, \Upsilon_k \leq \Upsilon_{k+1}\}.$$

Obviously, τ is a non-decreasing sequence such that $\tau(n) \to \infty$, as $n \to \infty$ and

$$\Upsilon_{\tau(n)} \leq \Upsilon_{\tau(n)+1}, \ \forall\ n \geq n_0.$$

From (3.10), we have

$$\begin{aligned}&\gamma_n^2[||A^*(Aw_{\tau(n)} - Bz_{\tau(n)}||^2) + ||B^*(Aw_{\tau(n)} - Bz_{\tau(n)})||^2] \\ &\leq ||x_{\tau(n)} - p||^2 + ||y_{\tau(n)} - q||^2 \\ &- [||x_{\tau(n)+1} - p||^2 + ||y_{\tau(n)+1} - p||^2] + \alpha_{\tau(n)}[||u-p||^2 + ||v-q||^2] \\ &\leq \alpha_{\tau(n)}[||u-p||^2 + ||v-q||^2].\end{aligned}$$

Hence,

$$\gamma^2_{\tau(n)}[||A^*(Aw_{\tau(n)} - Bz_{\tau(n)}||^2) + ||B^*(Aw_{\tau(n)} - Bz_{\tau(n)})||^2] \to 0, \text{ as } n \to \infty.$$

By the condition on $\{\gamma_{\tau(n)}\}$, we have

$$[||A^*(Aw_{\tau(n)} - Bz_{\tau(n)}||^2) + ||B^*(Aw_{\tau(n)} - Bz_{\tau(n)})||^2] \to 0, \text{ as } n \to \infty.$$

Note that $Aw_{\tau(n)} - Bz_{\tau(n)} = 0$, if $\tau(n) \notin \Omega$. Hence,

$$\lim_{n\to\infty} ||A^*(Aw_{\tau(n)} - Bz_{\tau(n)})||^2 = 0, \tag{3.38}$$

and

$$\lim_{n\to\infty} ||B^*(Aw_{\tau(n)} - Bz_{\tau(n)})||^2 = 0. \tag{3.39}$$

Now for all $n \geq n_0$, we have from (3.37) that

$$\begin{aligned} 0 &\leq [||x_{\tau(n)+1} - \overline{x}||^2 + ||y_{\tau(n)+1} - \overline{y}||^2 - (||x_{\tau(n)} - \overline{x}||^2 + ||y_{\tau(n)} - \overline{y}||^2)] \\ &\leq (1 - \alpha_{\tau(n)})[||x_{\tau(n)} - \overline{x}||^2 + ||y_{\tau(n)} - \overline{y}||^2] - [||x_{\tau(n)} - \overline{x}||^2 + ||y_{\tau(n)} - \overline{y}||^2] \\ &+ \alpha_{\tau(n)}[\alpha_{\tau(n)}[||u - \overline{x}||^2 + ||v - \overline{y}||^2] + 2(1 - \alpha_{\tau(n)})(\langle x_{\tau(n)} - \overline{x}, u - \overline{x}\rangle + \langle y_{\tau(n)} - \overline{y}, v - \overline{y}\rangle)], \end{aligned}$$

which implies

$$\begin{aligned} &||x_{\tau(n)} - \overline{x}||^2 + ||y_{\tau(n)} - \overline{y}||^2 \\ &\leq \alpha_{\tau(n)}[||u - \overline{x}||^2 + ||v - \overline{y}||^2] + 2(1 - \alpha_{\tau(n)})(\langle x_{\tau(n)} - \overline{x}, u - \overline{x}\rangle + \langle y_{\tau(n)} - \overline{y}, v - \overline{y}\rangle) \to 0. \end{aligned}$$

Hence,

$$\lim_{n\to\infty} (||x_{\tau(n)} - \overline{x}||^2 + ||y_{\tau(n)} - \overline{y}||^2) = 0.$$

Therefore,

$$\lim_{n\to\infty} \Upsilon_{\tau(n)} = \lim_{n\to\infty} \Upsilon_{\tau(n)+1} = 0.$$

Moreso, for $n \geq n_0$, it is clear that $\Upsilon_{\tau(n)} \leq \Upsilon_{\tau(n)+1}$ if $n \neq \tau(n)$ (*i.e* $\tau(n) < n$) because $\Upsilon_j > \Upsilon_{j+1}$ for $\tau(n) + 1 \leq j \leq n$.
Consequently for all $n \geq n_0$,

$$0 \leq \Upsilon_n \leq \max\{\Upsilon_{\tau(n)}, \Upsilon(n) + 1\} = \Upsilon_{\tau(n)+1}.$$

Therefore, we conclude that $\lim_{n\to\infty} \Upsilon_n = 0$, which also implies that $\{(x_n, y_n)\}$ converges strongly to $(\overline{x}, \overline{y})$. □

4. Numerical Example

Let $H = \mathbb{R}^2$ be endowed with the Euclidean norm and $T : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $T(x_1, x_2) = \frac{1}{3}(x_1, x_2)$ and $R(x_1, x_2) = \frac{2}{5}(x_1, x_2)$. Then T and R are nonexpansive mappings. Now, define $\psi : \mathbb{R}^2 \to (-\infty, \infty]$ by $\psi(x) = \frac{1}{2}||Bx - b||^2$, where $B(x) = (2x_1 + x_2, x_1 + 3x_2)$ and $b = (0, 0)$. Then f is a proper convex and lower semi-continuous function, since B is a continuous linear mapping.
Also, define $\varphi : \mathbb{R}^2 \to (-\infty, \infty]$ by $\varphi(x) = ||Mx - m||^2$ by $M(x) = (3x_1 - x_2, 2x_1 + 3x_2)$ and $m = (0, 0)$. Then φ is a proper convex and lower semi-continuous function. Let $\rho_n = \mu_n = 1 \; \forall \; n \geq 1$, then

$$\begin{aligned} J_1^{\psi}(x) &= argmin_{y\in\mathbb{R}^2}[\psi(y) + \frac{1}{2}||y - x||^2] = [I + B^T B]^{-1}(x + B^T b^T) \\ &= (\frac{11x_1 - 5x_2}{41}, \frac{-5x_1 + 6x_2}{41}). \end{aligned}$$

Also,

$$J_1^{\varphi}(x) = argmin_{y\in\mathbb{R}^2}[\varphi(y) + \frac{1}{2}||y-x||^2] = [I + M^T M]^{-1}(x + M^T m^T)$$
$$= (\frac{11x_1 - 3x_2}{145}, \frac{-3x_1 + 14x_2}{145}).$$

Now take $\alpha_n = \frac{1}{n+5}$ and $\beta_n = \frac{n}{2n+3}, \forall\ n \geq 1$.
Then (3.1) becomes

$$\begin{cases} w_n = \frac{n+4}{n+5}x_n + \frac{1}{n+5}u, \\ z_n = \frac{n+4}{n+5}y_n + \frac{1}{n+5}v, \\ u_n = J^{\psi}_{\rho_n}(w_n - \gamma_n A^*(Aw_n - Bz_n)), \\ v_n = J^{\varphi}_{\mu_n}(z_n + \gamma_n A^*(Aw_n - Bz_n)), \\ x_{n+1} = \frac{n+3}{2n+3}u_n + \frac{n}{6n+9}u_n, \\ y_{n+1} = \frac{n+3}{2n+3}v_n + \frac{2n}{10n+15}v_n \end{cases} \tag{4.1}$$

Let $A(x_1, x_2) = 2$ and $B(x_1, x_2) = 6x$, so that $A^*(x) = 2x$ and $B^*(x) = 6x$.
Let $\gamma_n \in \left(\epsilon, \frac{2||Aw_n - Bz_n||^2}{||A^*(Aw_n - Bz_n)||^2 + ||B^*(Aw_n - Bz_n)||^2} - \epsilon\right)$, $n \in \Omega$. Otherwise $\gamma_n = \gamma(\gamma$ being any any nonnegative value), where the set of indexes $\Omega = \{n : Aw_n - Bz_n \neq 0\}$.

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