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Well-posedness and exponential stability for a piezoelectric beams system with magnetic and thermal effects in the presence of past history

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Abstract

In this article, we consider the one-dimensional system of piezoelectric beams with thermal and magnetic effects in the presence of an infinite memory term acting on the mechanical equation. Under appropriate assumptions on the kernel, we prove that the system is well-posed in the sense of semigroup and by constructing a suitable Lyapunov functional. We establish that the system is exponentially stable. Moreover, our result does not depend on any relationship between system parameters.

Keywords: Piezoelectric beams, past history, semigroup approach, Lyapunov functional, energy method, exponential stability.
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1. Introduction

Piezoelectric materials such as quartz, Rochelle salt and barium titanate have an important property of converting mechanical energy to electro-magnetic energy under the action of a mechanical stress, this phenomenon is known by the direct piezoelectric effect that was discovered by the brothers Pierre and Jacques Curie in 1880. Reciprocally, the same materials have the ability to convert electro-magnetic energy to mechanical energy and this phenomena is well called the converse piezoelectric effect that was discovered by Gabriel Lippmann [27] in 1881. In addition, during the transformation of mechanical energy into electric one, it also turns a small portion of it into magnetic energy [19]. This last energy has a relatively small

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effect on the general dynamics and there exist the models that neglect magnetic effects such as piezoelectric beams. However, this magnetic contribution may limit the system performance, for example, the magnetic effect can cause oscillations in the output which leads to system instability in closed loop [23, 29]. In [19], by applying a variational approach, Morris and Özer constructed a coupled model of piezoelectric beams with magnetic effect given by

$$\begin{cases} \rho v_{tt} - \alpha v_{xx} + \gamma\beta p_{xx} = 0 \text{ in } (0,L)\times(0,\infty), \\ \mu p_{tt} - \beta p_{xx} + \gamma\beta v_{xx} = 0 \text{ in } (0,L)\times(0,\infty), \end{cases} \tag{1.1}$$

where ρ, α, γ, μ, β and L are positive constants represent, respectively, the mass density, elastic stiffness, piezoelectric coefficient, magnetic permeability, water resistance coefficient of the beam and the length of the beam. In addition, the relationship is considered

$$\alpha = \alpha_1 + \gamma^2\beta \text{ with } \alpha_1 > 0, \tag{1.2}$$

and the system (1.1) is equipped by the boundary and initial conditions

$$\begin{cases} v(0,t) = p(0,t) = \alpha v_x(L,t) - \gamma\beta p_x(L,t) = 0, \\ \beta p_x(L,t) - \gamma\beta v_x(L,t) = -\dfrac{V(t)}{h}, \\ v(x,0) = v_0(x), \ v_t(x,0) = v_1(x), \ p(x,0) = p_0(x), \ p_t(x,0) = p_1(x), \end{cases} \tag{1.3}$$

where $V(t)$ is the voltage applied at the electrode and h is the thickness of the beam. In [25], Ramos et al. investigated the piezoelectric beams system

$$\begin{cases} \rho v_{tt} - \alpha v_{xx} + \gamma\beta p_{xx} + \delta v_t = 0 \text{ in } (0,L)\times(0,T), \\ \mu p_{tt} - \beta p_{xx} + \gamma\beta v_{xx} = 0 \text{ in } (0,L)\times(0,T), \end{cases} \tag{1.4}$$

with the boundary and initial conditions

$$\begin{cases} v(0,t) = \alpha v_x(L,t) - \gamma\beta p_x(L,t) = 0, \ 0 \le t \le T, \\ p(0,t) = p_x(L,t) - \gamma v_x(L,t) = 0, \ 0 \le t \le T, \\ v(x,0) = v_0(x), \ v_t(x,0) = v_1(x), \ p(x,0) = p_0(x), \ p_t(x,0) = p_1(x), \ 0 \le x \le L, \end{cases}$$

and they showed that the dissipation given only by the magnetic effect is strong enough to stabilize exponentially the system for whatever the physical parameters of the model. In [24], Ramos et al. investigated the one-dimensional system of piezoelectric beams with magnetic effect given by

$$\begin{cases} \rho v_{tt} - \alpha v_{xx} + \gamma\beta p_{xx} = 0 \text{ in } (0,L)\times(0,T), \\ \mu p_{tt} - \beta p_{xx} + \gamma\beta v_{xx} = 0 \text{ in } (0,L)\times(0,T), \end{cases}$$

with the boundary conditions

$$\begin{cases} v(0,t) = \alpha v_x(L,t) - \gamma\beta p_x(L,t) + \xi_1 \frac{v_t(L,t)}{h} = 0, \ 0 < t < T, \\ p(0,t) = \beta p_x(L,t) - \gamma\beta v_x(L,t) + \xi_2 \frac{p_t(L,t)}{h} = 0, \ 0 < t < T, \end{cases}$$

where ξ_1 and ξ_2 are positive constants. The initial conditions are given by

$$v(x,0) = v_0(x), \ v_t(x,0) = v_1(x), \ p(x,0) = p_0(x), \ p_t(x,0) = p_1(x), \ \forall x \in (0,L).$$

They showed that the system is exponentially stable regardless of any condition on the coefficients of the system and exponential stability is equivalent to exact observability at the boundary. In [8], Freitas et al. investigated the piezoelectric beam system with thermal and magnetic effects and with friction damping

$$\begin{cases} \rho v_{tt} - \alpha v_{xx} + \gamma\beta p_{xx} + \delta\theta_x + f_1(v,p) = h_1 \text{ in } (0,L)\times(0,T), \\ \mu p_{tt} - \beta p_{xx} + \gamma\beta v_{xx} + A^v p_t + f_2(v,p) = h_2 \text{ in } (0,L)\times(0,T), \\ c\theta_t - \kappa\theta_{xx} + \delta v_{tx} = 0 \text{ in } (0,L)\times(0,T), \end{cases} \tag{1.5}$$

with the boundary and initial conditions

$$\begin{cases} v(0,t) = \alpha v_x(L,t) - \gamma\beta p_x(L,t) = 0, \ t > 0, \\ p(0,t) = p_x(L,t) - \gamma v_x(L,t) = 0, \ t > 0, \\ \theta(0,t) = \theta(L,t) = 0, \ t > 0, \\ v(x,0) = v_0(x), \ v_t(x,0) = v_1(x), \ p(x,0) = p_0(x), \ 0 < x < L, \\ p_t(x,0) = p_1(x), \ \theta(x,0) = \theta_0(x), \ 0 < x < L, \end{cases} \tag{1.6}$$

where the physical constants ρ, α, β, γ, δ, κ, μ and c are positive, θ is a temperature difference, f_1, f_2 are nonlinear source terms and h_1, h_2 are external forces. They proved that the dynamical system generated by the problem (1.5) and (1.6) has a smooth global attractor. Other problems related to piezoelectric systems can be found in the following references [4, 7, 17, 18, 28]. Motivated by the above works, in this paper, we consider the following problem

$$\begin{cases} \rho v_{tt} - \alpha v_{xx} + \gamma\beta p_{xx} + \delta\theta_x + \int_0^\infty g(s)\, v_{xx}(t-s)\, ds = 0 \text{ in } (0,L)\times(0,\infty), \\ \mu p_{tt} - \beta p_{xx} + \gamma\beta v_{xx} = 0 \text{ in } (0,L)\times(0,\infty), \\ c\theta_t - \kappa\theta_{xx} + \delta v_{tx} = 0 \text{ in } (0,L)\times(0,\infty). \end{cases} \tag{1.7}$$

This system is subjected to the boundary and initial conditions

$$\begin{cases} v(x,0) = v_0(x), \ v_t(x,0) = v_1(x), \ p(x,0) = p_0(x), \ x \in (0,L), \\ p_t(x,0) = p_1(x), \ \theta(x,0) = \theta_0(x), \ x \in (0,L), \\ v(0,t) = v_x(L,t) = p(0,t) = p_x(L,t) = \theta(0,t) = \theta(L,t) = 0, \ t \in (0,\infty), \end{cases}$$

where the integral in the infinite memory term can be regarded as a natural weak damping term, the function g is called the relaxation function, the initial data v_0, v_1, p_0, p_1 and θ_0 are specified later. The purpose of this paper is to prove a exponential decay estimate for solutions of the system (1.7). Moreover, our results depend on the construction of a suitable Lyapunov functional and the kernel of the infinite memory term which allows us to estimate the energy of the system. In the presence of this complementary control, the main problem concerning the stability is determining the largest class of kernels g which guarantee the stability and the best relation between the solutions of the considered system and the decay rates. However, it remains with great importance in the study of the asymptotic behavior of the solution for the different types of problems that can be found in the references [3, 6, 10, 11, 12, 13, 14, 15, 21, 26, 30].

The article is organized as follows. We present some assumptions and transformations in Section 2. We prove the existence and uniqueness result of solutions of (1.7) applying the semigroup technique in section 3. We demonstrate that the system is exponentially stable in section 4.

2. Preliminaries

To prove our main result, in this section we present the backgrounds mathematics needed later. We shall apply the following hypotheses. For the memory kernel $g = g(s)$, we assume that

(H1) The function g satisfying

$$g \in C^1(\mathbb{R}_+) \cap L^1(\mathbb{R}_+), \ g(s) > 0, \ \forall s \in \mathbb{R}_+, \ \alpha_1 - g_0 = l > 0, \ g_0 = \int_0^\infty g(s) ds. \tag{2.1}$$

(H2) There exist two positive constants δ_0 and δ_1 such that

$$-\delta_0 g(s) \le g'(s) \le -\delta_1 g(s), \ \forall s \in \mathbb{R}_+, \tag{2.2}$$

with $\delta_1 g_0 \le g(0) \le \delta_0 g_0$.

As noted in [5], (2.2) implies that $g(s)$ decays exponentially, so

$$\lim_{s\to\infty} g(s) = 0. \tag{2.3}$$

Remark 2.1. The assumption (2.2) is a very natural inequality and can be found in several works, for example [1, 20].

Lemma 2.2 ([12])**.** *The next inequalities hold,*

$$\int_0^L \left(\int_0^\infty g(s) \left(v(t) - v(t-s) \right) ds \right)^2 dx \leq d_1 \left(g \circ v_x \right)(t), \tag{2.4}$$

$$\int_0^L \left(\int_0^\infty g'(s) \left(v_x(t) - v_x(t-s) \right) ds \right)^2 dx \leq -g(0) \left(g' \circ v_x \right)(t), \tag{2.5}$$

$$\int_0^L \left(\int_0^\infty g(s) \left(v_x(t) - v_x(t-s) \right) ds \right)^2 dx \leq g_0 \left(g \circ v_x \right)(t), \tag{2.6}$$

$$\int_0^L \left(\int_0^\infty g'(s) \left(v(t) - v(t-s) \right) ds \right)^2 dx \leq -d_2 (g' \circ v_x)(t), \tag{2.7}$$

where d_1, d_2 are positive constants and

$$(g \circ \nu)(t) = \int_0^L \int_0^\infty g(s)(\nu(x,t) - \nu(x,t-s))^2 ds dx.$$

Here are some notations that will help us for the computation of energy

$$\eta^t(x,s) = v(x,t) - v(x,t-s), \ (x,t,s) \in (0,L) \times \mathbb{R}_+ \times \mathbb{R}_+,$$

which was adopted in articles [20, 21], and η^t is the relative history of v satisfies

$$\begin{cases} \eta_t^t + \eta_s^t - v_t = 0, \ (x,t,s) \in (0,L) \times \mathbb{R}_+ \times \mathbb{R}_+, \\ \eta^t(0,s) = 0, \ \eta_x^t(L,s) = 0, \ t,s > 0, \\ \eta^t(x,0) = 0, \ \eta^0(x,s) = \eta_0(x,s), \ x \in (0,L), \ t,s > 0, \end{cases} \tag{2.8}$$

where the functional class of η_0 is given by the functional class of the initial data. Now, in order to deal with the variable η^t in the memory term, we introduce the following weighted Hilbert space

$$L_g := L_g^2\left(\mathbb{R}_+, \tilde{H}^1(0,L)\right) = \left\{ \varphi : \mathbb{R}_+ \longrightarrow \tilde{H}^1(0,L), \ \int_0^L \left(\int_0^\infty g(s) \varphi_x^2 ds \right) dx < \infty \right\},$$

where

$$\tilde{H}^1(0,L) = \left\{ f \in H^1(0,L) : f(0) = 0 \right\},$$

The space L_g is endowed with the next inner product

$$\langle \varphi_1, \varphi_2 \rangle_{L_g} = \int_0^L \int_0^\infty g(s) \varphi_{1x} \varphi_{2x} ds dx.$$

We now consider the linear operator $\mathcal{T}$, defined on L_g and given by

$$\mathcal{T}\varphi := -\varphi_s, \ \forall \varphi \in D(\mathcal{T}),$$

where φ_s is derivative of φ in distributional sense with respect to s, and

$$D(\mathcal{T}) := \{ \varphi \in L_g, \ \varphi_s \in L_g, \ \varphi(0) = 0 \}.$$

The operator $\mathcal{T}$ is the infinitesimal generator of a C_0-semigroup of contractions (see [9]). By integrating by parts and by taking into account (2.2) and (2.3), we get

$$\begin{aligned}
\langle \mathcal{T}\varphi, \varphi \rangle_{L_g} &= -\int_0^L \int_0^\infty g(s)\, \varphi_{sx}\varphi_x ds dx = -\int_0^\infty g(s) \int_0^L \varphi_{sx}\varphi_x dx ds \\
&= -\frac{1}{2}\int_0^\infty g(s) \frac{\partial}{\partial s}\left(\int_0^L \varphi_x^2 dx\right) ds \\
&= \left[-\frac{1}{2} g(s) \int_0^L \varphi_x^2 dx\right]_0^\infty + \frac{1}{2}\int_0^\infty g'(s) \int_0^L \varphi_x^2 dx ds \\
&= \frac{1}{2}\int_0^L \int_0^\infty g'(s)\, \varphi_x^2 ds dx \le -\frac{\delta_1}{2}\int_0^L \int_0^\infty g(s)\varphi_x^2 ds dx = -\frac{\delta_1}{2}\left\|\varphi\right\|_{L_g}^2.
\end{aligned}$$

Then, by the introduction of η^t in the system (1.7), the system (1.7) is equivalent to

$$\begin{cases}
\rho v_{tt} - \alpha v_{xx} + \gamma\beta p_{xx} + \delta\theta_x + \int_0^\infty g(s)\, v_{xx}(t-s)\, ds = 0 \text{ in } (0,L)\times(0,\infty), \\
\mu p_{tt} - \beta p_{xx} + \gamma\beta v_{xx} = 0 \text{ in } (0,L)\times(0,\infty), \\
c\theta_t - \kappa\theta_{xx} + \delta v_{tx} = 0 \text{ in } (0,L)\times(0,\infty), \\
\eta_t^t + \eta_s^t - v_t = 0, \ (x,t,s) \in (0,L)\times\mathbb{R}_+\times\mathbb{R}_+, \\
v(0,t) = v_x(L,t) = p(0,t) = p_x(L,t) = \theta(0,t) = \theta(L,t) = 0, \ t \in (0,\infty), \\
\eta^t(0,s) = \eta_x^t(L,s) = 0, \ t,s \in (0,\infty), \\
v(x,0) = v_0(x), \ v_t(x,0) = v_1(x), \ x \in (0, L), \\
p(x,0) = p_0(x), \ p_t(x,0) = p_1(x), \ \theta(x,0) = \theta_0(x), \ x \in (0, L), \\
\eta^t(x,0) = 0, \ \eta^0(x,s) = \eta_0(x,s), \ x \in (0,L), \ t,s \in (0,\infty),
\end{cases} \tag{2.9}$$

3. The Well-Posedness of the Problem

In this section, we give the existence and uniqueness of solutions for the system (2.9) applying semigroup theory. [16, 22]. First, we introduce the vector function $U = (v, u, p, q, \theta, \eta^t)^T$, with $u = v_t$ and $q = p_t$. The first equation of (2.9) can be rewritten as follows

$$\rho v_{tt} - \left(l + \gamma^2\beta\right) v_{xx} + \gamma\beta p_{xx} + \delta\theta_x - \int_0^\infty g(s)\, \eta_{xx}^t(x,s)\, ds = 0. \tag{3.1}$$

Therefore, the system (2.9) can be rewritten as the following form

$$\begin{cases}
\frac{dU}{dt} + \mathcal{A}U = 0, \ t > 0, \\
U(x,0) = U_0(x) = (v_0, v_1, p_0, p_1, \theta_0, \eta_0)^T,
\end{cases} \tag{3.2}$$

where the operator $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \to \mathcal{H}$ is the linear operator defined by

$$\mathcal{A} = \begin{pmatrix}
0 & -I & 0 & 0 & 0 & 0 \\
-\frac{(l+\gamma^2\beta)}{\rho}\partial_{xx} & 0 & \frac{\gamma\beta}{\rho}\partial_{xx} & 0 & \frac{\delta}{\rho}\partial_x & -\frac{1}{\rho}\int_0^\infty g(s)\,\partial_{xx} ds \\
0 & 0 & 0 & -I & 0 & 0 \\
\frac{\gamma\beta}{\mu}\partial_{xx} & 0 & -\frac{\beta}{\mu}\partial_{xx} & 0 & 0 & 0 \\
0 & \frac{\delta}{c}\partial_x & 0 & 0 & -\frac{\kappa}{c}\partial_{xx} & 0 \\
0 & -I & 0 & 0 & 0 & -\mathcal{T}
\end{pmatrix},$$

where

$$\mathcal{A}U = \begin{pmatrix}
-u \\
-\frac{(l+\gamma^2\beta)}{\rho} v_{xx} + \frac{\gamma\beta}{\rho} p_{xx} + \frac{\delta}{\rho}\theta_x - \frac{1}{\rho}\int_0^\infty g(s)\eta_{xx}^t ds \\
-q \\
\frac{\gamma\beta}{\mu} v_{xx} - \frac{\beta}{\mu} p_{xx} \\
\frac{\delta}{c} u_x - \frac{\kappa}{c}\theta_{xx} \\
-\mathcal{T}\eta^t - u
\end{pmatrix},$$

and $\mathcal{H}$ is the energy space given by

$$\mathcal{H}=\tilde{H}^1(0,L)\times L^2(0,L)\times \tilde{H}^1(0,L)\times L^2(0,L)\times L^2(0,L)\times L_g,$$

such that

$$\tilde{H}^1(0,L)=\left\{f\in H^1(0,L): f(0)=0\right\},$$
$$\tilde{H}^2(0,L)=H^2(0,L)\cap \tilde{H}^1(0,L).$$

Then $\mathcal{H}$, along with the inner product

$$\left\langle U,\tilde{U}\right\rangle_{\mathcal{H}}=\rho\int_0^L u\tilde{u}dx+\mu\int_0^L q\tilde{q}dx+c\int_0^L \theta\widetilde{\theta}dx+l\int_0^L v_x\tilde{v}_xdx$$
$$+\beta\int_0^L(\gamma v_x-p_x)(\gamma\tilde{v}_x-\tilde{p}_x)\,dx+\left\langle \eta^t,\tilde{\eta}^t\right\rangle_{L_g}, \tag{3.3}$$

is a Hilbert space for any $U=(v,u,p,q,\theta,\eta^t)^T\in\mathcal{H}$ and $\tilde{U}=(\tilde{v},\tilde{u},\tilde{p},\tilde{q},\widetilde{\theta},\tilde{\eta}^t)^T\in\mathcal{H}$. The domain of $\mathcal{A}$ is given by

$$D(\mathcal{A})=\Big\{U\in\mathcal{H}: v,p\in\tilde{H}^2(0,L),\ u,q\in\tilde{H}^1(0,L),\ \theta\in H^2(0,L)\cap H_0^1(0,L),$$
$$\eta^t\in D(\mathcal{T}),\ \int_0^\infty g(s)\eta^t_{xx}ds\in L^2(0,L),\ v_x(L)=p_x(L)=0\Big\}.$$

Clearly, if $\mathcal{A}$ is a maximal monotone operator, then $D(\mathcal{A})$ is dense in $\mathcal{H}$.

Definition 3.1. A bilinear form $B:\mathcal{H}\times\mathcal{H}\to\mathbb{R}$ is said to be coercive if there is a constant $\varrho>0$ such that

$$B(v,v)\geq\varrho\|v\|^2_{\mathcal{H}},\ \forall v\in\mathcal{H}.$$

Lemma 3.2 (Lax-Milgram [2])**.** *Let $\mathcal{H}$ be a Hilbert space equipped with the norm $\|\cdot\|_{\mathcal{H}}$. Let B be a continuous coercive bilinear form on $\mathcal{H}$ and $\mathcal{G}$ be a continuous linear form on $\mathcal{H}$, then there exists a unique $u\in\mathcal{H}$ such that*

$$B(u,v)=\mathcal{G}(v),\ \forall v\in\mathcal{H}.$$

Now, we can give the next existence result.

Theorem 3.3. *Let $U_0\in\mathcal{H}$ and assume that (H1)–(H2) holds. Then, there exists a unique solution $U\in C(\mathbb{R}_+,\mathcal{H})$ for problem (3.2). Moreover, if $U_0\in D(\mathcal{A})$, then*

$$U\in C(\mathbb{R}_+,D(\mathcal{A}))\cap C^1(\mathbb{R}_+,\mathcal{H}).$$

Proof. We apply the semigroup approach. Sufficiently, we show that $\mathcal{A}$ is a maximal monotone operator. First, we prove that $\mathcal{A}$ is monotone. For any $U\in D(\mathcal{A})$, applying integration by parts, we have

$$\langle\mathcal{A}U,U\rangle_{\mathcal{H}}=\left\langle\begin{pmatrix}-u\\ -\frac{(l+\gamma^2\beta)}{\rho}v_{xx}+\frac{\gamma\beta}{\rho}p_{xx}+\frac{\delta}{\rho}\theta_x-\frac{1}{\rho}\int_0^\infty g(s)\eta^t_{xx}ds\\ -q\\ \frac{\gamma\beta}{\mu}v_{xx}-\frac{\beta}{\mu}p_{xx}\\ \frac{\delta}{c}u_x-\frac{\kappa}{c}\theta_{xx}\\ -\mathcal{T}\eta^t-u\end{pmatrix},\begin{pmatrix}v\\ u\\ p\\ q\\ \theta\\ \eta^t\end{pmatrix}\right\rangle_{\mathcal{H}},$$

we obtain

$$\langle\mathcal{A}U,U\rangle_{\mathcal{H}}=-\left\langle\mathcal{T}\eta^t,\eta^t\right\rangle_{L_g}+\kappa\int_0^L\theta_x^2dx\geq 0.$$

Thus, $\mathcal{A}$ is monotone.

Next, we demonstrate that the operator $I+\mathcal{A}$ is surjective. Given $\mathcal{F}=(f_1,f_2,f_3,f_4,f_5,f_6)^T\in\mathcal{H}$, we show that there exists a unique $U\in D(\mathcal{A})$ such that

$$(I+\mathcal{A})\,U=\mathcal{F}. \tag{3.4}$$

That is,

$$\begin{cases} v-u=f_1\in\tilde{H}^1(0,L)\,,\\ \rho u-\left(l+\gamma^2\beta\right)v_{xx}+\gamma\beta p_{xx}+\delta\theta_x-\int_0^\infty g(s)\eta_{xx}^t ds=\rho f_2\in L^2(0,L)\,,\\ p-q=f_3\in\tilde{H}^1(0,L)\,,\\ \mu q-\beta p_{xx}+\gamma\beta v_{xx}=\mu f_4\in L^2(0,L)\,,\\ c\theta-\kappa\theta_{xx}+\delta u_x=cf_5\in L^2(0,L)\,,\\ \eta^t-\mathcal{T}\eta^t-u=f_6\in L_g. \end{cases} \tag{3.5}$$

Using $(3.5)_6$, we obtain

$$\eta^t(s)=\left(1-e^{-s}\right)u+e^{-s}\int_0^s e^\tau f_6(\tau)\,d\tau. \tag{3.6}$$

Inserting $u=v-f_1$ in $(3.5)_2$, $(3.5)_5$, $q=p-f_3$ in $(3.5)_4$ and (3.6) in $(3.5)_2$, we obtain

$$\begin{cases} \rho v-\left[\left(l+\gamma^2\beta\right)+\int_0^\infty g(s)\left(1-e^{-s}\right)ds\right]v_{xx}+\gamma\beta p_{xx}+\delta\theta_x=h_1\in L^2(0,L)\,,\\ \mu p-\beta p_{xx}+\gamma\beta v_{xx}=J_1\in L^2(0,L)\,,\\ c\theta-\kappa\theta_{xx}+\delta v_x=Q\in L^2(0,L)\,, \end{cases} \tag{3.7}$$

where

$$\begin{cases} h_1=\rho(f_1+f_2)+\int_0^\infty g(s)e^{-s}\int_0^s e^\tau(f_6-f_1)_{xx}\,d\tau ds,\\ J_1=\mu(f_3+f_4)\,,\ Q=cf_5+\delta f_{1x}. \end{cases}$$

To solve (3.7), we consider the next variational formulation

$$B\left((v,p,\theta),(v_1,p_1,\theta_1)\right)=\mathcal{G}(v_1,p_1,\theta_1)\,, \tag{3.8}$$

where $B:\left[\tilde{H}^1(0,L)\times\tilde{H}^1(0,L)\times H_0^1(0,L)\right]^2\longrightarrow\mathbb{R}$ is the bilinear form defined by

$$\begin{aligned} B\left((v,p,\theta),(v_1,p_1,\theta_1)\right)=&\ \rho\int_0^L vv_1dx+\left[l+\int_0^\infty g(s)\left(1-e^{-s}\right)ds\right]\int_0^L v_xv_{1x}dx\\ &+\mu\int_0^L pp_1dx+\beta\int_0^L(\gamma v_x-p_x)(\gamma v_{1x}-p_{1x})\,dx\\ &+\delta\int_0^L(\theta_xv_1+v_x\theta_1)\,dx+c\int_0^L\theta\theta_1dx+\kappa\int_0^L\theta_x\theta_{1x}dx, \end{aligned}$$

and $\mathcal{G}:\left[\tilde{H}^1(0,L)\times\tilde{H}^1(0,L)\times H_0^1(0,L)\right]\longrightarrow\mathbb{R}$ is the linear functional given by

$$\mathcal{G}(v_1,p_1,\theta_1)=\int_0^L h_1v_1dx+\int_0^L J_1p_1dx+\int_0^L Q\theta_1dx.$$

Now, for $W=\tilde{H}^1(0,L)\times\tilde{H}^1(0,L)\times H_0^1(0,L)$ equipped with the norm

$$\|(v,p,\theta)\|_W^2=\|v\|_2^2+\|v_x\|_2^2+\|p\|_2^2+\|(\gamma v_x-p_x)\|_2^2+\|\theta\|_2^2+\|\theta_x\|_2^2\,,$$

applying integration by parts, we have

$$\begin{aligned} B\left((v,p,\theta),(v,p,\theta)\right)=&\ \rho\|v\|_2^2+\left[l+\int_0^\infty g(s)\left(1-e^{-s}\right)ds\right]\|v_x\|_2^2+\mu\|p\|_2^2\\ &+\beta\|(\gamma v_x-p_x)\|_2^2+c\|\theta\|_2^2+\kappa\|\theta_x\|_2^2\,. \end{aligned}$$

Then, for some $M_0 > 0$

$$B\left(\left(v,p,\theta\right),\left(v,p,\theta\right)\right) \geq M_0 \left\|\left(v,p,\theta\right)\right\|_W^2.$$

So, B is coercive. On the other hand, applying the Cauchy-Schwarz inequality, we get

$$\left|B\left(\left(v,p,\theta\right),\left(v_1,p_1,\theta_1\right)\right)\right| \leq n_1 \left\|\left(v,p,\theta\right)\right\|_W \left\|\left(v_1,p_1,\theta_1\right)\right\|_W.$$

Similarly

$$\left|\mathcal{G}\left(v_1,p_1,\theta_1\right)\right| \leq n_2 \left\|\left(v_1,p_1,\theta_1\right)\right\|_W.$$

So, by applying the Lax-Milgram Lemma, we demonstrate the existence of a unique

$$(v,p,\theta) \in \tilde{H}^1\left(0,L\right) \times \tilde{H}^1\left(0,L\right) \times H_0^1\left(0,L\right),$$

satisfying

$$B\left(\left(v,p,\theta\right),\left(v_1,p_1,\theta_1\right)\right) = \mathcal{G}\left(v_1,p_1,\theta_1\right), \quad \forall\left(v_1,p_1,\theta_1\right) \in W.$$

By through $(3.5)_1$, we have $u - v \in \tilde{H}^1\left(0,L\right)$. So, $u = v + u - v \in \tilde{H}^1\left(0,L\right)$. By a similar way, the substitution of p into $(3.5)_3$ yields $q \in \tilde{H}^1\left(0,L\right)$. Hence

$$(u,q) \in \tilde{H}^1\left(0,L\right) \times \tilde{H}^1\left(0,L\right). \tag{3.9}$$

Now, to show $U = (v,u,p,q,\theta,\eta^t)^T \in D\left(\mathcal{A}\right)$ we will prove that $\eta^t \in D\left(\mathcal{T}\right)$. The function given in (3.6), satisfies $(3.5)_6$, with $\eta^t(0) = 0$. Moreover, the second term on the right side in (3.6)

$$s \in \mathbb{R}_+ \longmapsto e^{-s}\int_0^s e^{\tau} f_6\left(\tau\right)d\tau,$$

belongs to L_g. This can be seen by changing the order within the integral

$$\begin{aligned}
&\int_0^\infty g\left(s\right)\int_0^L \left(e^{-s}\int_0^s e^{\tau} f_{6x}\left(\tau\right)d\tau\right)^2 dxds \\
&= \int_0^\infty g\left(s\right)e^{-2s}\int_0^L \left(\int_0^s e^{\tau} f_{6x}\left(\tau\right)d\tau\right)^2 dxds \\
&\leq \int_0^\infty g\left(s\right)e^{-2s}\int_0^L \left(\int_0^s e^{\tau} d\tau\right)\left(\int_0^s e^{\tau}\left|f_{6x}\left(\tau\right)\right|^2 d\tau\right) dxds \\
&\leq \int_0^\infty g\left(s\right)e^{-s}\int_0^s e^{\tau}\left(\int_0^L \left|f_{6x}\left(\tau\right)\right|^2 dx\right) d\tau ds \\
&= \int_0^\infty \int_\tau^\infty g\left(s\right)e^{-s}e^{\tau}\left(\int_0^L \left|f_{6x}\left(\tau\right)\right|^2 dx\right) dsd\tau \\
&= \int_0^\infty e^{\tau}\left(\int_0^L \left|f_{6x}\left(\tau\right)\right|^2 dx\right)\left(\int_\tau^\infty g\left(s\right)e^{-s}ds\right) d\tau \\
&\leq \int_0^\infty g\left(\tau\right)\left(\int_0^L \left|f_{6x}\left(\tau\right)\right|^2 dx\right) d\tau \\
&= \int_0^L \int_0^\infty g\left(\tau\right)\left|f_{6x}\left(\tau\right)\right|^2 d\tau dx < \infty.
\end{aligned}$$

Moreover, since $u \in \tilde{H}^1\left(0,L\right)$, we can deduce that $\eta^t \in L_g$ and as

$$\eta_s^t\left(s\right) = e^{-s}u - e^{-s}\int_0^s e^{\tau} f_6\left(\tau\right)d\tau + f_6\left(s\right),$$

we also have $\eta_s^t \in L_g$ and this way $\eta^t \in D(\mathcal{T})$. Moreover, if we take $(p_1,\theta_1)=(0,0)\in \tilde{H}^1(0,L)\times H_0^1(0,L)$ in (3.8), then we obtain

$$\begin{cases} \rho\int_0^L vv_1dx + \left[\left(l+\gamma^2\beta\right)+\int_0^\infty g(s)\left(1-e^{-s}\right)ds\right]\int_0^L v_xv_{1x}dx \\ -\gamma\beta\int_0^L p_xv_{1x}dx + \delta\int_0^L \theta_x v_1dx = \int_0^L h_1v_1dx,\ \forall v_1\in\tilde{H}^1(0,L). \end{cases} \tag{3.10}$$

Multiplying $(3.7)_2$ by γ and adding with $(3.7)_1$, we obtain

$$v_{xx} = \frac{\rho v + \mu\gamma p + \delta\theta_x - h_1 - \gamma J_1}{\left[l+\int_0^\infty g(s)\left(1-e^{-s}\right)ds\right]} \in L^2(0,L). \tag{3.11}$$

Consequently, we obtain

$$v\in\tilde{H}^2(0,L).$$

Furthermore, if we take $(v_1,\theta_1)=(0,0)\in\tilde{H}^1(0,L)\times H_0^1(0,L)$ in (3.8), then we obtain

$$\mu\int_0^L pp_1dx + \beta\int_0^L p_xp_{1x}dx - \gamma\beta\int_0^L v_xp_{1x}dx = \int_0^L J_1p_1dx,\ \forall p_1\in\tilde{H}^1(0,L). \tag{3.12}$$

By exploiting $(3.7)_2$ and (3.11), we obtain

$$p_{xx} = \gamma v_{xx} + \frac{\mu}{\beta}p - \frac{1}{\beta}J_1 \in L^2(0,L), \tag{3.13}$$

Consequently, we obtain

$$p\in\tilde{H}^2(0,L).$$

Similarly, if we take $(v_1,p_1)=(0,0)\in\tilde{H}^1(0,L)\times\tilde{H}^1(0,L)$ in (3.8), then we have

$$c\int_0^L \theta\theta_1dx + \kappa\int_0^L \theta_x\theta_{1x}dx + \delta\int_0^L v_x\theta_1dx = \int_0^L Q\theta_1dx,\ \forall\theta_1\in H_0^1(0,L). \tag{3.14}$$

By exploiting $(3.7)_3$, we obtain

$$\theta_{xx} = \frac{c}{\kappa}\theta + \frac{\delta}{\kappa}v_x - \frac{1}{\kappa}Q \in L^2(0,L). \tag{3.15}$$

Consequently, we obtain

$$\theta\in H^2(0,L)\cap H_0^1(0,L).$$

Now, by applying $(3.5)_2$ and exploiting (3.9), (3.11), (3.13), (3.15), then we get

$$\int_0^\infty g(s)\eta_{xx}^t ds = \rho u - \left(l+\gamma^2\beta\right)v_{xx} + \gamma\beta p_{xx} + \delta\theta_x - \rho f_2 \in L^2(0,L).$$

Consequently, we obtain

$$\int_0^\infty g(s)\eta_{xx}^t ds \in L^2(0,L).$$

Thus, by integrating (3.12) and (3.10) by parts and exploiting $(3.7)_1$, $(3.7)_2$, then we obtain

$$\begin{cases} \left(\beta p_x(L)-\gamma\beta v_x(L)\right)p_1(L) - \left(\beta p_x(0)-\gamma\beta v_x(0)\right)p_1(0)=0, \\ \left(\left[\left(l+\gamma^2\beta\right)+\int_0^\infty g(s)\left(1-e^{-s}\right)ds\right]v_x(L)-\gamma\beta p_x(L)\right)v_1(L) \\ -\left(\left[\left(l+\gamma^2\beta\right)+\int_0^\infty g(s)\left(1-e^{-s}\right)ds\right]v_x(0)-\gamma\beta p_x(0)\right)v_1(0)=0. \end{cases}$$

Furthermore, if we take $p_1=\frac{\gamma x}{L}$ and $v_1=\frac{x}{L}$, then we get

$$\begin{cases} \gamma\beta p_x(L)-\gamma^2\beta v_x(L)=0, \\ \left[\left(l+\gamma^2\beta\right)+\int_0^\infty g(s)\left(1-e^{-s}\right)ds\right]v_x(L)-\gamma\beta p_x(L)=0. \end{cases} \tag{3.16}$$

By performing some calculations on the above expression (3.16), we get

$$\left[l+\int_0^{\infty} g(s)\left(1-e^{-s}\right) ds\right] v_x(L)=0,$$

and as $\left[l+\int_0^{\infty} g(s)\left(1-e^{-s}\right) ds\right]>0$, then we obtain

$$v_x(L)=0. \tag{3.17}$$

By substituting (3.17) into (3.16), then we get

$$p_x(L)=0.$$

Therefore,

$$v_x(L)=p_x(L)=0.$$

Then, there is a unique $U \in D(\mathcal{A})$ such that (3.4) is satisfied. Hence, $\mathcal{A}$ is a maximal monotone operator. So, by applying the Hille-Yosida theorem, we obtain the well-posedness result. □

4. Exponential stability

In this section, we prove and state the technical lemmas needed for the proof of our stability results.

Lemma 4.1. *Let (v,p,θ) be a solution of (2.9). Then, the energy functional $E(t)$, defined by*

$$E(t)=\frac{1}{2}\int_0^L\left[\rho v_t^2+\mu p_t^2+l v_x^2+\beta\left(\gamma v_x-p_x\right)^2+c\theta^2\right] dx+\frac{1}{2}\left(g\circ v_x\right)(t), \tag{4.1}$$

satisfies,

$$E'(t)=-\kappa\int_0^L \theta_x^2 dx+\frac{1}{2}\left(g'\circ v_x\right)(t)\leq-\kappa\int_0^L \theta_x^2 dx-\frac{\delta_1}{2}\left(g\circ v_x\right)(t)\leq 0. \tag{4.2}$$

Proof. Multiplying (3.1), $(2.9)_2$ and $(2.9)_3$ by v_t, p_t and θ respectively, integrating over $(0,L)$, taking into account the boundary conditions and summing them up, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_0^L\left[\rho v_t^2+\mu p_t^2+l v_x^2+\beta\left(\gamma v_x-p_x\right)^2+c\theta^2\right] dx\\
&+\kappa\int_0^L \theta_x^2 dx-\int_0^L v_t\left(\int_0^{\infty} g(s)\,\eta_{xx}^t(x,s)\, ds\right) dx \;=\; 0.
\end{aligned} \tag{4.3}$$

We estimate the last term of (4.3) as follows

$$\begin{aligned}
&-\int_0^L v_t\left(\int_0^{\infty} g(s)\eta_{xx}^t(x,s)ds\right) dx\\
&=-\int_0^L\left(\eta_t^t+\eta_s^t\right)\left(\int_0^{\infty} g(s)\eta_{xx}^t(x,s)ds\right) dx\\
&=-\int_0^{\infty} g(s)\left(\int_0^L \eta_t^t\eta_{xx}^t(x,s)dx\right) ds-\int_0^{\infty} g(s)\left(\int_0^L \eta_s^t\eta_{xx}^t(x,s)dx\right) ds.
\end{aligned}$$

Integrating by parts, we have

$$-\int_0^L v_t\left(\int_0^{\infty} g(s)\eta_{xx}^t(x,s)ds\right) dx=\frac{1}{2}\frac{d}{dt}\left(g\circ v_x\right)(t)-\frac{1}{2}\left(g'\circ v_x\right)(t). \tag{4.4}$$

By substituting (4.4) into (4.3), bearing in mind (4.1), yields (4.2). □

Lemma 4.2. *Let (v, p, θ) be a solution of (2.9). Then, the functional*

$$I_1(t) = \rho \int_0^L v_t v dx + \gamma\mu \int_0^L v p_t dx,\ t \geq 0,$$

satisfies for any $\varepsilon_1 > 0$,

$$I_1'(t) \leq -\frac{l}{4} \int_0^L v_x^2 dx + \varepsilon_1 \int_0^L p_t^2 dx + \left(\rho + \frac{\gamma^2\mu^2}{4\varepsilon_1}\right) \int_0^L v_t^2 dx + \frac{\delta^2 c}{l} \int_0^L \theta_x^2 dx + \frac{g_0}{2l} \left(g \circ v_x\right)(t),\ \forall t \geq 0. \tag{4.5}$$

Proof. By differentiating $I_1(t)$, applying $(2.9)_1$, $(2.9)_2$ and integrating by parts together with the boundary conditions, we obtain

$$I_1'(t) = -\alpha_1 \int_0^L v_x^2 dx + \rho \int_0^L v_t^2 dx + \gamma\mu \int_0^L p_t v_t dx - \delta \int_0^L \theta_x v dx + \int_0^L v_x \left(\int_0^\infty g(s) \nu_x(x, t-s)\, ds\right) dx. \tag{4.6}$$

The Young inequality leads to

$$\gamma\mu \int_0^L p_t v_t dx \leq \varepsilon_1 \int_0^L p_t^2 dx + \frac{\gamma^2\mu^2}{4\varepsilon_1} \int_0^L v_t^2 dx, \tag{4.7}$$

$$-\delta \int_0^L \theta_x v dx \leq \frac{l}{4} \int_0^L v_x^2 dx + \frac{\delta^2 c}{l} \int_0^L \theta_x^2 dx, \tag{4.8}$$

and

$$\begin{aligned}
&\int_0^L v_x \left(\int_0^\infty g(s)\, v_x(x, t-s)\, ds\right) dx \\
&= -\int_0^L v_x \left(\int_0^\infty g(s) \left(v_x(x,t) - v_x(x, t-s)\right) ds\right) dx + g_0 \int_0^L v_x^2 dx \\
&\leq (\delta_1 + g_0) \int_0^L v_x^2 dx + \frac{1}{4\delta_1} \int_0^L \left(\int_0^\infty g(s) \left(v_x(x,t) - v_x(x,t-s)\right) ds\right)^2 dx \\
&\leq (\delta_1 + g_0) \int_0^L v_x^2 dx + \frac{g_0}{4\delta_1} \left(g \circ v_x\right)(t),
\end{aligned} \tag{4.9}$$

Substituting (4.7), (4.8) and (4.9) in (4.6) and letting $\delta_1 = \frac{l}{2}$, we get (4.5). □

Lemma 4.3. *Let (v, p, θ) be a solution of (2.9). Then, the functional*

$$I_2(t) = \int_0^L \left(\rho v_t + \gamma\mu p_t\right) \left(\gamma v - p\right) dx,\ t \geq 0,$$

satisfies, for any $\varepsilon_2 > 0$,

$$I_2'(t) \leq -\frac{\gamma\mu}{2} \int_0^L p_t^2 dx + 4\varepsilon_2 \int_0^L \left(\gamma v_x - p_x\right)^2 dx + \frac{1}{4\varepsilon_2} \left(g_0^2 + \alpha_1^2\right) \int_0^L v_x^2 dx + \left(\gamma\rho + \frac{\left(\gamma^2\mu - \rho\right)^2}{2\gamma\mu}\right) \int_0^L v_t^2 dx + \frac{\delta^2 c}{4\varepsilon_2} \int_0^L \theta_x^2 dx + \frac{g_0}{4\varepsilon_2} \left(g \circ v_x\right)(t). \tag{4.10}$$

Proof. By differentiating $I_2(t)$, applying $(2.9)_1$, $(2.9)_2$ and integrating by parts together with the boundary conditions, we obtain

$$\begin{aligned}I_2'(t) &= -\alpha_1 \int_0^L v_x \left(\gamma v_x - p_x\right) dx - \delta \int_0^L \theta_x \left(\gamma v - p\right) dx \\ &+ \left(\gamma^2 \mu - \rho\right) \int_0^L v_t p_t dx + \gamma\rho \int_0^L v_t^2 dx \\ &- \gamma\mu \int_0^L p_t^2 dx + \int_0^L \left(\gamma v_x - p_x\right) \left(\int_0^\infty g(s)\, v_x(x, t-s)\, ds\right) dx. \end{aligned} \tag{4.11}$$

Using the Young inequality, we get

$$- \alpha_1 \int_0^L v_x \left(\gamma v_x - p_x\right) dx \le \varepsilon_2 \int_0^L \left(\gamma v_x - p_x\right)^2 dx + \frac{\alpha_1^2}{4\varepsilon_2} \int_0^L v_x^2 dx, \tag{4.12}$$

$$- \delta \int_0^L \theta_x \left(\gamma v - p\right) dx \le \varepsilon_2 \int_0^L \left(\gamma v_x - p_x\right)^2 dx + \frac{\delta^2 c}{4\varepsilon_2} \int_0^L \theta_x^2 dx, \tag{4.13}$$

$$\left(\gamma^2 \mu - \rho\right) \int_0^L v_t p_t dx \le \frac{\gamma\mu}{2} \int_0^L p_t^2 dx + \frac{\left(\gamma^2\mu - \rho\right)^2}{2\gamma\mu} \int_0^L v_t^2 dx. \tag{4.14}$$

Using the fact that

$$\begin{aligned}&\int_0^L \left(\gamma v_x - p_x\right) \left(\int_0^\infty g(s)\, v_x(x, t-s)\, ds\right) dx \\ &= - \int_0^L \left(\gamma v_x - p_x\right) \left(\int_0^\infty g(s) \left(v_x(x,t) - v_x(x, t-s)\right) ds\right) dx + g_0 \int_0^L \left(\gamma v_x - p_x\right) v_x dx.\end{aligned}$$

By applying the Young inequality again, we have

$$\begin{aligned}&- \int_0^L \left(\gamma v_x - p_x\right) \left(\int_0^\infty g(s) \left(v_x(x,t) - v_x(x, t-s)\right) ds\right) dx \\ &\le \varepsilon_2 \int_0^L \left(\gamma v_x - p_x\right)^2 dx + \frac{g_0}{4\varepsilon_2} \left(g \circ v_x\right)(t), \end{aligned} \tag{4.15}$$

and

$$g_0 \int_0^L \left(\gamma v_x - p_x\right) v_x dx \le \varepsilon_2 \int_0^L \left(\gamma v_x - p_x\right)^2 dx + \frac{g_0^2}{4\varepsilon_2} \int_0^L v_x^2 dx. \tag{4.16}$$

Inserting (4.12)–(4.16) in (4.11), we obtain (4.10). □

Lemma 4.4. *Let (v, p, θ) be a solution of (2.9). Then, the functional*

$$I_3(t) = \rho \int_0^L v v_t dx + \mu \int_0^L p p_t dx,\ \ t \ge 0,$$

satisfies

$$\begin{aligned}I_3'(t) &\le -\frac{l}{4} \int_0^L v_x^2 dx - \beta \int_0^L \left(\gamma v_x - p_x\right)^2 dx + \mu \int_0^L p_t^2 dx \\ &+ \rho \int_0^L v_t^2 dx + \frac{\delta^2 c}{l} \int_0^L \theta_x^2 dx + \frac{g_0}{2l} \left(g \circ v_x\right)(t). \end{aligned} \tag{4.17}$$

Proof. By differentiating $I_3(t)$, applying $(2.9)_1$, $(2.9)_2$ and integrating by parts together with the boundary conditions, we obtain

$$\begin{aligned} I_3'(t) &= -\beta \int_0^L (\gamma v_x - p_x)^2 \, dx + \rho \int_0^L v_t^2 dx - \alpha_1 \int_0^L v_x^2 dx - \delta \int_0^L \theta_x v dx \\ &+ \mu \int_0^L p_t^2 dx + \int_0^L v_x \left(\int_0^\infty g(s) \, v_x(x, t-s) \, ds \right) dx. \end{aligned} \tag{4.18}$$

Using the Young inequality, we arrive at

$$\begin{aligned} &\int_0^L v_x \left(\int_0^\infty g(s) \, v_x(x, t-s) \, ds \right) dx \\ &= -\int_0^L v_x \left(\int_0^\infty g(s) \left(v_x(x,t) - v_x(x, t-s) \right) ds \right) dx + g_0 \int_0^L v_x^2 dx \\ &\leq (\delta_1 + g_0) \int_0^L v_x^2 dx + \frac{1}{4\delta_1} \int_0^L \left(\int_0^\infty g(s) \left(v_x(x,t) - v_x(x,t-s) \right) ds \right)^2 dx \\ &\leq (\delta_1 + g_0) \int_0^L v_x^2 dx + \frac{g_0}{4\delta_1} \left(g \circ v_x \right)(t), \end{aligned} \tag{4.19}$$

and

$$-\delta \int_0^L \theta_x v dx \leq \frac{l}{4} \int_0^L v_x^2 dx + \frac{\delta^2 c}{l} \int_0^L \theta_x^2 dx. \tag{4.20}$$

Substituting (4.19) and (4.20) into (4.18) and letting $\delta_1 = \dfrac{l}{2}$, we get (4.17). □

Lemma 4.5. *Let (v, p, θ) be a solution of (2.9). Then, the functional*

$$I_4(t) = -\frac{\rho}{\gamma} \int_0^L v_t \left(\int_0^\infty g(s)(v(x,t) - v(x,t-s)) ds \right) dx,$$

satisfies, for any $\varepsilon_3, \varepsilon_4 > 0$, the following estimate

$$\begin{aligned} I_4'(t) &\leq -\frac{\rho g_0}{2\gamma} \int_0^L v_t^2 dx + 2\varepsilon_3 \int_0^L v_x^2 dx + \varepsilon_4 \int_0^L (\gamma v_x - p_x)^2 \, dx \\ &+ \int_0^L \theta_x^2 dx + C_{\varepsilon_3, \varepsilon_4} (g \circ v_x)(t), \end{aligned} \tag{4.21}$$

where

$$C_{\varepsilon_3,\varepsilon_4} = \left(\frac{\left(\alpha - \gamma^2 \beta\right)^2 g_0}{4\varepsilon_3 \gamma^2} + \frac{\beta^2 g_0}{4\varepsilon_4} + \frac{g_0}{\gamma} \left(1 + \frac{g_0^2}{4\varepsilon_3 \gamma} \right) + \frac{\delta^2 d_1}{4\gamma^2} + \frac{\rho d_2 \delta_0}{2\gamma g_0} \right).$$

Proof. First, we have

$$\begin{aligned} &\frac{\partial}{\partial t} \left(\int_0^\infty g(s)(v(x,t) - v(x,t-s)) ds \right) \\ &= \frac{\partial}{\partial t} \left(\int_{-\infty}^t g(t-s)(v(x,t) - v(x,s)) ds \right) \\ &= \int_{-\infty}^t g'(t-s)(v(x,t) - v(x,s)) ds + \int_{-\infty}^t g(t-s) v_t(x,t) ds \\ &= g_0 v_t + \int_0^\infty g'(s)(v(x,t) - v(x,t-s)) ds. \end{aligned}$$

By differentiating $I_4(t)$, applying $(2.9)_1$ and integrating by parts, we get

$$
\begin{aligned}
I_4'(t) = & \left(\frac{\alpha}{\gamma}-\gamma\beta\right)\int_0^L v_x\left(\int_0^\infty g(s)(v_x(x,t)-v_x(x,t-s))ds\right)dx \\
& +\beta\int_0^L(\gamma v_x-p_x)\left(\int_0^\infty g(s)(v_x(x,t)-v_x(x,t-s))ds\right)dx \\
& +\frac{\delta}{\gamma}\int_0^L\theta_x\left(\int_0^\infty g(s)(v(x,t)-v(x,t-s))ds\right)dx \\
& -\frac{1}{\gamma}\int_0^L\left(\int_0^\infty g(s)\,v_x(x,t-s)\,ds\right)\left(\int_0^\infty g(s)(v_x(x,t)-v_x(x,t-s))ds\right)dx \\
& -\frac{\rho g_0}{\gamma}\int_0^L v_t^2dx-\frac{\rho}{\gamma}\int_0^L v_t\left(\int_0^\infty g'(s)(v(x,t)-v(x,t-s))ds\right)dx. \qquad (4.22)
\end{aligned}
$$

Using the Young inequality, (2.6) and (2.7), we have

$$
\begin{aligned}
& \left(\frac{\alpha}{\gamma}-\gamma\beta\right)\int_0^L v_x\left(\int_0^\infty g(s)(v_x(x,t)-v_x(x,t-s))ds\right)dx \qquad (4.23)\\
& \leq \varepsilon_3\int_0^L v_x^2dx+\frac{\left(\alpha-\gamma^2\beta\right)^2 g_0}{4\varepsilon_3\gamma^2}(g\circ v_x)(t),
\end{aligned}
$$

$$
\begin{aligned}
& -\frac{\rho}{\gamma}\int_0^L v_t\left(\int_0^\infty g'(s)(v(x,t)-v(x,t-s))ds\right)dx \\
& \leq \frac{\rho g_0}{2\gamma}\int_0^L v_t^2dx-\frac{\rho d_2}{2\gamma g_0}(g'\circ v_x)(t) \\
& \leq \frac{\rho g_0}{2\gamma}\int_0^L v_t^2dx+\frac{\rho d_2\delta_0}{2\gamma g_0}(g\circ v_x)(t), \qquad (4.24)
\end{aligned}
$$

$$
\begin{aligned}
& \frac{\delta}{\gamma}\int_0^L\theta_x\left(\int_0^\infty g(s)(v(x,t)-v(x,t-s))ds\right)dx \\
& \leq \int_0^L\theta_x^2dx+\frac{\delta^2 d_1}{4\gamma^2}(g\circ v_x)(t), \qquad (4.25)
\end{aligned}
$$

$$
\begin{aligned}
& \beta\int_0^L(\gamma v_x-p_x)\left(\int_0^\infty g(s)(v_x(x,t)-v_x(x,t-s))ds\right)dx \\
& \leq \varepsilon_4\int_0^L(\gamma v_x-p_x)^2dx+\frac{\beta^2 g_0}{4\varepsilon_4}(g\circ v_x)(t), \qquad (4.26)
\end{aligned}
$$

and

$$
\begin{aligned}
& -\frac{1}{\gamma}\int_0^L\left(\int_0^\infty g(s)\,v_x(x,t-s)\,ds\right)\left(\int_0^\infty g(s)(v_x(x,t)-v_x(x,t-s))ds\right)dx \\
& =\frac{1}{\gamma}\int_0^L\left(\int_0^\infty g(s)(v_x(x,t)-v_x(x,t-s))ds\right)^2dx \\
& -\frac{g_0}{\gamma}\int_0^L v_x(x,t)\left(\int_0^\infty g(s)(v_x(x,t)-v_x(x,t-s))ds\right)dx \\
& \leq \frac{g_0}{\gamma}\left(1+\frac{g_0^2}{4\varepsilon_3\gamma}\right)(g\circ v_x)(t)+\varepsilon_3\int_0^L v_x^2dx. \qquad (4.27)
\end{aligned}
$$

Estimate (4.21) follows by substituting (4.23)–(4.27) into (4.22). □

Now, we define the Lyapunov functional $\mathcal{L}(t)$ by

$$\mathcal{L}(t) = NE(t) + N_1 I_1(t) + N_2 I_2(t) + N_3 I_3(t) + N_4 I_4(t), \tag{4.28}$$

where N, N_1, N_2, N_3 and N_4 are positive constants.

Lemma 4.6. *Let (v, p, θ) be a solution of (2.9). Then, there exist two positive constants κ_1 and κ_2 such that the Lyapunov functional (4.28) satisfies*

$$\kappa_1 E(t) \le \mathcal{L}(t) \le \kappa_2 E(t), \ \forall t \ge 0, \tag{4.29}$$

and

$$\mathcal{L}'(t) \le -\beta_1 E(t), \ \forall t \ge 0. \tag{4.30}$$

Proof. From (4.28), we have

$$\begin{aligned}
|\mathcal{L}(t) - NE(t)| \le{}& \rho N_1 \int_0^L |v_t v| \, dx + \gamma\mu N_1 \int_0^L |v p_t| \, dx \\
&+ N_2 \int_0^L |(\rho v_t + \gamma\mu p_t)| \, |(\gamma v - p)| \, dx \\
&+ \rho N_3 \int_0^L |v v_t| \, dx + \mu N_3 \int_0^L |p p_t| \, dx \\
&+ \frac{\rho N_4}{\gamma} \int_0^L |v_t| \left(\int_0^\infty g(s) \, |(v(x,t) - v(x,t-s))| \, ds \right) dx.
\end{aligned}$$

By applying the Young, Poincaré, Cauchy-Schwarz inequalities and the hypothesis (2.4), we obtain

$$|\mathcal{L}(t) - NE(t)| \le \tau E(t),$$

which yields

$$(N - \tau) E(t) \le \mathcal{L}(t) \le (N + \tau) E(t),$$

by choosing N (depending on N_1, N_2, N_3 and N_4) sufficiently large we obtain (4.29). Now, By differentiating $\mathcal{L}(t)$, using Lemma 4.1 to Lemma 4.5, we get

$$\begin{aligned}
\mathcal{L}'(t) \le{}& -\left[\frac{l}{4}(N_3 + N_1) - \frac{1}{4\varepsilon_2}\left(g_0^2 + \alpha_1^2\right) N_2 - 2\varepsilon_3 N_4 \right] \int_0^L v_x^2 dx \\
&- \left[\frac{\gamma\mu}{2} N_2 - \varepsilon_1 N_1 - \mu N_3 \right] \int_0^L p_t^2 dx \\
&- \left[\frac{\rho g_0}{2\gamma} N_4 - \left(\rho + \frac{\gamma^2\mu^2}{4\varepsilon_1} \right) N_1 - \left(\gamma\rho + \frac{\left(\gamma^2\mu - \rho\right)^2}{2\gamma\mu} \right) N_2 - \rho N_3 \right] \int_0^L v_t^2 dx \\
&- \left[\beta N_3 - 4\varepsilon_2 N_2 - \varepsilon_4 N_4 \right] \int_0^L (\gamma v_x - p_x)^2 \, dx \\
&- \left[\kappa N - \frac{\delta^2 c}{l}(N_1 + N_3) - N_2 \frac{\delta^2 c}{4\varepsilon_2} - N_4 \right] \int_0^L \theta_x^2 dx \\
&- \left[\frac{N}{2}\delta_1 - \frac{g_0}{2l}(N_1 + N_3) - \frac{g_0}{4\varepsilon_2} N_2 - C_{\varepsilon_3,\varepsilon_4} N_4 \right] (g \circ v_x)(t).
\end{aligned} \tag{4.31}$$

By setting $\varepsilon_1 = \frac{1}{N_1}$, $\varepsilon_2 = \frac{1}{N_2}$, $\varepsilon_3 = \varepsilon_4 = \frac{1}{N_4}$,

$$\begin{aligned}\mathcal{L}'(t) \leq & -\left[\frac{l}{4}(N_1+N_3) - \frac{1}{4}\left(g_0^2+\alpha_1^2\right)N_2^2 - 2\right]\int_0^L v_x^2 dx \\ & -\left[\frac{\gamma\mu}{2}N_2 - \mu N_3 - 1\right]\int_0^L p_t^2 dx \\ & -\left[\frac{\rho g_0}{2\gamma}N_4 - \left(\rho + \frac{\gamma^2\mu^2}{4}N_1\right)N_1 - \left(\gamma\rho + \frac{\left(\gamma^2\mu-\rho\right)^2}{2\gamma\mu}\right)N_2 - \rho N_3\right]\int_0^L v_t^2 dx \\ & -\left[\beta N_3 - 5\right]\int_0^L \left(\gamma v_x - p_x\right)^2 dx \\ & -\left[\kappa N - \frac{\delta^2 c}{l}(N_1+N_3) - N_2^2\frac{\delta^2 c}{4} - N_4\right]\int_0^L \theta_x^2 dx \\ & -\left[\frac{N}{2}\delta_1 - \frac{g_0}{2l}(N_1+N_3) - \frac{g_0}{4}N_2^2 - CN_4\right](g\circ v_x)(t).\end{aligned}$$

Now, we select our parameters appropriately as follows. First, we choose N_3 large enough so that

$$\delta = \beta N_3 - 5 > 0.$$

Next, we select N_2 large enough so that

$$\delta_2 = \frac{\gamma\mu}{2}N_2 - 1 - \mu N_3 > 0.$$

We take N_1 large such that

$$\delta_3 = \frac{l}{4}(N_1+N_3) - \frac{1}{4}\left(g_0^2+\alpha_1^2\right)N_2^2 - 2 > 0.$$

We select N_4 large enough so that

$$\delta_4 = \frac{\rho g_0}{2\gamma}N_4 - \left(\rho + \frac{\gamma^2\mu^2}{4}N_1\right)N_1 - \left(\gamma\rho + \frac{\left(\gamma^2\mu-\rho\right)^2}{2\gamma\mu}\right)N_2 - \rho N_3 > 0.$$

Finally, we choose N large enough so that (4.29) remains valid, further

$$\begin{aligned}\delta_5 &= \kappa N - \frac{\delta^2 c}{l}(N_1+N_3) - N_2^2\frac{\delta^2 c}{4} - N_4 > 0, \\ \delta_6 &= \frac{N}{2}\delta_1 - \frac{g_0}{2l}(N_1+N_3) - \frac{g_0}{4}N_2^2 - CN_4 > 0.\end{aligned}$$

So, we end up with

$$\begin{aligned}\mathcal{L}'(t) \leq & -\delta\int_0^L \left(\gamma v_x - p_x\right)^2 dx - \delta_2\int_0^L p_t^2 dx - \delta_3\int_0^L v_x^2 dx \\ & -\delta_4\int_0^L v_t^2 dx - \delta_5\int_0^L \theta_x^2 dx - \delta_6(g\circ v_x)(t).\end{aligned}$$

Use the Poincaré inequality to substitute $-\int_0^L \theta_x^2 dx$ by $-\int_0^L \theta^2 dx$, we get

$$\begin{aligned}\mathcal{L}'(t) \leq & -\delta\int_0^L \left(\gamma v_x - p_x\right)^2 dx - \delta_2\int_0^L p_t^2 dx - \delta_3\int_0^L v_x^2 dx \\ & -\delta_4\int_0^L v_t^2 dx - c\delta_5\int_0^L \theta^2 dx - \delta_6(g\circ v_x)(t) \\ \leq & -\varpi\left[\int_0^L \left[v_x^2 + p_t^2 + \left(\gamma v_x - p_x\right)^2 + v_t^2 + \theta^2\right]dx + (g\circ v_x)(t)\right], \end{aligned} \tag{4.32}$$

where $\varpi = \min\left(\delta, \delta_2, \delta_3, \delta_4, c\delta_5, \delta_6\right) > 0$. On the other hand, we have

$$E(t) \leq c\left[\int_0^L \left[v_x^2 + p_t^2 + (\gamma v_x - p_x)^2 + v_t^2 + \theta^2\right] dx + (g \circ v_x)(t)\right],$$

which implies that

$$-\left[\int_0^L \left[v_x^2 + p_t^2 + (\gamma v_x - p_x)^2 + v_t^2 + \theta^2\right] dx + (g \circ v_x)(t)\right] \leq -c' E(t). \tag{4.33}$$

The combination of (4.32) and (4.33) gives (4.30). □

Now, we will use the equivalence relation (4.29) to estimate the energy of (2.9) by applying the estimation (4.30). Hence, we can state and prove the next stability result.

Theorem 4.7. *Let (v, p, θ) be a solution of (2.9). Then, the solution (v, p, θ) decays exponentially, i.e. there exist two positive constants λ_1 and λ_2 such that*

$$E(t) \leq \lambda_2 e^{-\lambda_1 t}, \ \forall t \geq 0. \tag{4.34}$$

Proof. By applying Lemma 4.6, we get

$$\mathcal{L}'(t) \leq -\beta_1 E\left(t\right), \ \forall t \geq 0. \tag{4.35}$$

By exploiting the equivalence relation (4.29), we infer that

$$-\beta_1 E\left(t\right) \leq -\frac{\beta_1}{\kappa_2}\mathcal{L}(t), \ \forall t \geq 0. \tag{4.36}$$

By substituting (4.36) into (4.35), we get

$$\mathcal{L}'(t) \leq -\lambda_1 \mathcal{L}\left(t\right), \ \forall t \geq 0, \tag{4.37}$$

where $\lambda_1 = \beta_1/\kappa_2 > 0$. A simple integration of (4.37) gives

$$\mathcal{L}(t) \leq \mathcal{L}(0)e^{-\lambda_1 t}, \ \forall t \geq 0.$$

By applying the other side of the equivalence relation (4.29) i.e. $\kappa_1 E\left(t\right) \leq \mathcal{L}(t)$, we obtain

$$E\left(t\right) \leq \lambda_2 e^{-\lambda_1 t}, \ \forall t \geq 0,$$

where $\lambda_2 = \mathcal{L}(0)/\kappa_1 > 0$. The proof is complete. □

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