Available online at http://jprm.sms.edu.pk/
Journal of Prime Research in Mathematics, 20(1) (2024), 43–57

Journal of Prime Research in Mathematics

# Optimizing Synchronization: A Comparative Study of Four-Dimensional Forced SIR Systems Employing Multiple Control Techniques

Derbouche Assia[a,*], Hamri Nasr-Eddine[a], Laouira Widad[a]

[a] *Laboratory of Mathematics and their Interactions, Abdelhafid Boussouf University Center, Mila, Algeria.*

**Abstract**

The forced SIR system's synchronization is the main subject of this study. Through the use of several control techniques, including active control (AC), active backstepping control (ABC), adaptive control (AdC), and sliding mode control (SMC). We redesigned the three-dimensional system to be autonomous and made it four-dimensional to ease numerical computations. The system's phase portrait, Lyapunov exponent graph, and bifurcation diagram are used to analyze its dynamic properties through numerical simulation. The effectiveness of the AC, SMC, ABC, and AdC approaches is examined using dynamical error and necessary control inputs. By contrasting the integral square error with the required control energy measurements, the best control for synchronization is found.

*Keywords:* SIR model, chaotic system, active control, sliding mode control, adaptive control, backstepping control, synchronization
*2010 MSC:* 37G35, 65P20, 34H10.

## 1. Introduction

For the past thirty years, chaos synchronization has been extensively studied to understand the behavior of complex systems and improve predictability. This research has led to significant advancements in various fields, demonstrating the practical implications of chaotic synchronization beyond theoretical interest. The idea of synchronization first appeared in the research of [16], who showed that two chaotic trajectories might converge. Afterward, many investigations have focused on the synchronization of chaotic systems in different contexts, such as secure communication [1], biological systems [28], finance [12], and engineering [3]. These studies have revealed the potential applications of chaotic synchronization in enhancing data

*Corresponding author
*Email addresses:* a.derbouche@centre-univ-mila.dz (Derbouche Assia), n.hamri@centre-univ-mila.dz (Hamri Nasr-Eddine), w.laouira@centre-univ-mila.dz (Laouira Widad)

Received: 23 April 2024; Accepted: 10 June 2024; Published Online: 12 June 2024.

encryption, modeling physiological processes, and designing robust control strategies. Numerous control techniques are used in chaotic synchronization, including linear control[26, 32], adaptive control (AdC) [9], sliding mode control (SMC)[4], active backstepping control (ABC)[24], and active control (AC)[14]. These control techniques are essential for synchronizing chaotic systems, providing stability and robustness. Every method presents both advantages and disadvantages. They are making it essential to choose the most appropriate one considering the particular dynamics of the system. This study focuses on one of the most basic epidemiological models, the SIR model. It was first presented in the work of [13]. Applications of this model can be found in epidemiology (COVID-19 [5], tuberculosis [20]), immunology, and public health policy. One of the most fundamental ideas in epidemiological systems is synchronization. When people engage in synchronization, it can impact the spread of infectious illnesses by speeding up the dissemination of germs. By focusing preventative efforts on synchronized populations, management techniques may be designed more effectively with an understanding of synchronization. But there are still things about synchronization that we don't fully understand, such as human behavior, societal variables, and the true effect on the transmission of illness. As well as we can also find this model in other fields, for example: as [15] points out, the SIR models can be solved precisely on various networks. [10] Provide an example of how the banking industry uses the SIR model. This model simulates the dynamics and behavior of credit default swap or CDS, markets. Researchers can better understand these markets' interconnectedness and potential risks by applying the SIR model to financial systems. This can help develop strategies to mitigate financial crises and improve market stability. Many research works focus on the model's qualitative study, related to bifurcations and equilibrium points in [25].[29] are curious about whether the forced SIR model has a periodic solution. [2] sought to determine the degree to which seasonality's effects affected epidemic models and analytically demonstrated the presence of topological horseshoes. The Lyapunov spectrum is computed in [8]'s work; this allows us to determine the fractal dimension, discern between different chaotic states, and assess the degree of predictability in the system's attractor appearance. The original nonautonomous SIR has a series of solutions we can acquire using the homotopy analysis method. In [6] work, whose attention is focused on the existence of strange attractors on the forced SIR system, [17], they provide a theoretical optimal control problem of social separation and show that a period-doubling-like phenomenon could arise. Our focus in this paper is on the forced SIR system's synchronization by applying the previously mentioned technique. The three-dimensional system was transformed into a four-dimensional one and then reconstructed as an autonomous system. The optimal controller for model synchronization is also selected by a comparative performance analysis. The remainder of the paper follows this structure: In section 1, the forced SIR system is described, providing a theoretical foundation for the control strategies discussed in the next sections. Section 2, details the scheme of AC, SMC, ABC, and AdC control methods, offering a comprehensive overview of each approach's implementation. Section 3 presents the analysis of each control method's performance in comparison to traditional control techniques. Finally, section 4 summarizes the main conclusions of the work and makes recommendations for future directions for study in the domain of infectious disease modeling and control.

## 2. Preliminaries

The error $e(t)$ must tend to zero as $t$ tend to infinity to achieve synchronization. This indicates that the error system should be stable. As a result, dynamic system theory and control techniques are frequently used to examine the error system's stability. This might involve Lyapunov stability assessments.If the error system is asymptotically stable, then e(t) converges toward zero, thus ensuring that the two chaotic systems synchronize, there is an extensive literature on the stability theory of differential equations of the third and fourth order such that[31], [32],[33], and [34] . The definitions listed below are required for this.
We consider the general nonlinear autonomous dynamical system

$$\begin{cases} \dot{x} = f(t, x) \\ x(0) = x_0 \end{cases} \tag{2.1}$$

Where $x \in \mathbb{R}^n$ et $f : \mathbb{R}^n \to \mathbb{R}^n$.

**Definition 2.1.** The equilibrium point $x^*$ of system (2.1)is called locally asymptotically stable if

$$\forall \varepsilon > 0,\, \exists \delta_\varepsilon > 0; \|x^* - x_0\| < \delta_\varepsilon \Rightarrow \lim_{t\to+\infty} \|x(t) - x^*\| = 0$$

**Definition 2.2.** We consider the system $\dot{x} = f(x)$. The total derivative or orbital derivative of $V$ in the direction of the vector field $f$ is defined as:

$$\frac{dV}{dt} = \frac{\partial x}{\partial t}\frac{\partial L}{\partial x} = f(x)\frac{\partial L}{\partial x}$$

**Definition 2.3.** Let $D \subset \mathbb{R}^n$ be an open neighborhood of the equilibrium point $x^*$. Then the function $V : D \to \mathbb{R}^n$, satisfying the following properties:

(i) $V$ is continuously differentiable,
(ii) $V > 0$ for all $x \in D - \{x^*\}$ and $V(x^*) = 0$

is called a Lyapunov function.

**Theorem 2.4.** *(Lyapunov theorem)*
*Suppose that the origin is an equilibrium point of $\dot{x} = f(x)$ with $x \in \mathbb{R}^n$ and let $V$ be a Lyapunov function in a neighborhood $D$ of the origin. If*

*(i) The orbital derivative $V \leq 0$ in $D$, that is, if $V$ is negative semi-definite in $D$ , the origin is stable,*
*(ii) $V < 0$ in $D - \{0\}$, that is, if $V$ negative definite in $D$, then the origin is asymptotically stable,*
*(iii) $V > 0$, that is, positive definite in $D$, the origin is unstable.*

## 3. Main results

### *3.1. Description of the forced SIR system*

The following differential equations describe the SIR system:

$$\begin{cases} \dot{S} &= \alpha - \phi(t)SI - \lambda S \\ \dot{I} &= \phi(t)SI - (\delta+\lambda)I \\ \dot{R} &= \delta I - \lambda R. \end{cases} \tag{3.1}$$

Where $S$ represents susceptibles, $I$ represents the infected persons, and $R$ represents the recovered persons. Shown by [7], the transmission rate fluctuates periodically with a period of $T$.

$$\phi(t) = \phi_0(1 + \beta \sin(\frac{2\pi}{T}t)) \tag{3.2}$$

The birth rate is typically $\alpha$, the death rate is $\lambda$, and the recovery rate is $\delta$. As in $\lambda \ll \delta$ in actuality, we assume that $\alpha = \lambda$, or that the birth and mortality rates are equal and that the babies are susceptible. Additionally, we assume that the population is sized such that $S + I + R = 1$ and has a constant density. As noted [8], the system (3.1) exhibits chaotic behavior when the parameter values are as follows. $\alpha = 0.01, \lambda = 0.01, \delta = 50, \phi_0 = 1510, T = 1,\ \beta \in [0.0960; 0.1073]$ and the states of the system take initials values $(S_0, I_0, R_0) = (0.03319, 0.00015, 0.96666)$. The two equilibria points are $E_1(1, 0, 0)$ and $E_2(0.03; 0.0002.0.97)$. The Lyapunov exponents are calculated as $L_{p1} = 0.2535$, $L_{p2} = 0.0286$ and $L_{p3} = -0.6918.L_{p2} > 0$, it means that the SIR system exhibits chaotic behavior.
In the rest of the paper, let's define $S = x_1, I = x_2, R = x_3$. Therefore, the system becomes:

$$\begin{cases} \dot{x_1} &= \alpha - \phi_0(1 + \beta\sin(\frac{2\pi}{T}t))x_1x_2 - \lambda x_1 \\ \dot{x_2} &= \phi_0(1 + \beta\sin(\frac{2\pi}{T}t))x_1x_2 - (\delta+\lambda)x_2 \\ \dot{x_3} &= \delta x_2 - \lambda x_3. \end{cases} \tag{3.3}$$

Where $(x_1, x_2, x_3)$ are the state variables. We transformed the three-dimensional system into a four-dimensional one and redesigned it to be autonomous. By letting $x_4 = \frac{2\pi}{T}t$ which implies $\dot{x_4} = \frac{2\pi}{T}$. Therefore, the new system is:

$$\begin{cases} \dot{x_1} &= \alpha - \phi_0(1+\beta\sin(x_4))x_1x_2 - \lambda x_1 \\ \dot{x_2} &= \phi_0(1+\beta\sin(x_4))x_1x_2 - (\delta+\lambda)x_2 \\ \dot{x_3} &= \delta x_2 - \lambda x_3 \\ \dot{x_4} &= \frac{2\pi}{T}. \end{cases} \tag{3.4}$$

The Lyapunov exponents are calculated as: $L_{p1} = 0.2063, L_{p2} = 0.0209, L_{p3} = -0.5368, L_{p4} = 0.000$

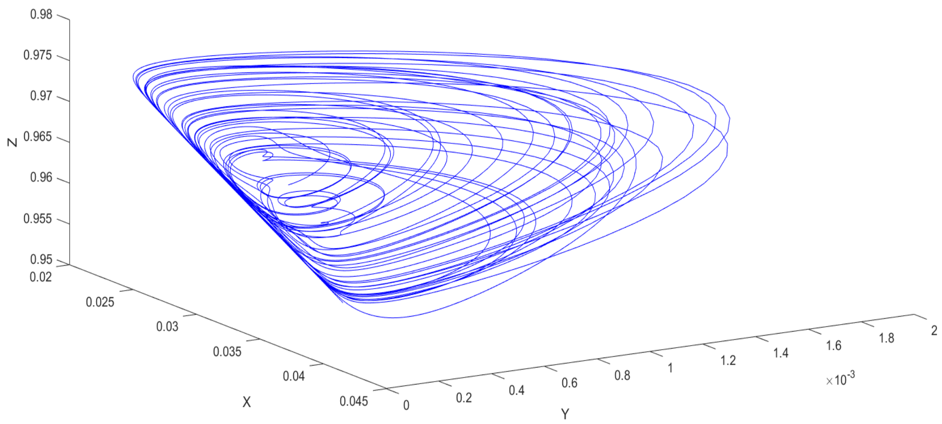


Figure 1: SIR attractor.

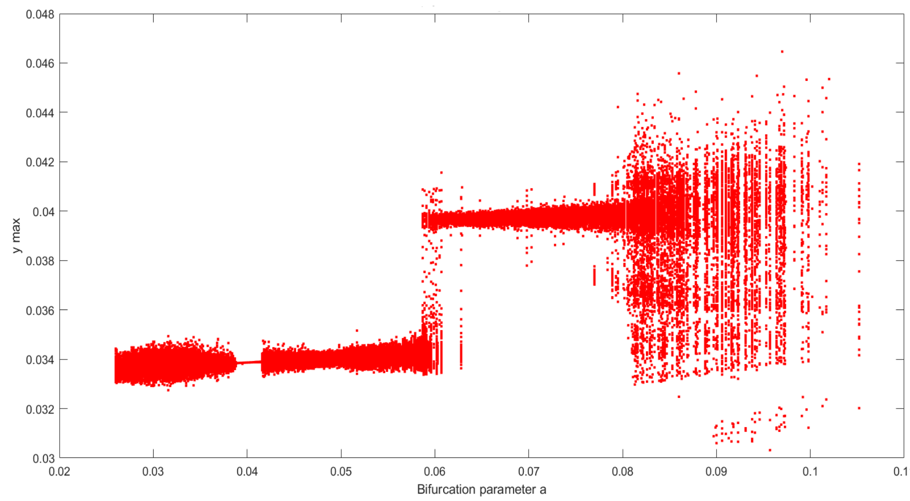


Figure 2: Bifurcation diagram of system (3.4).

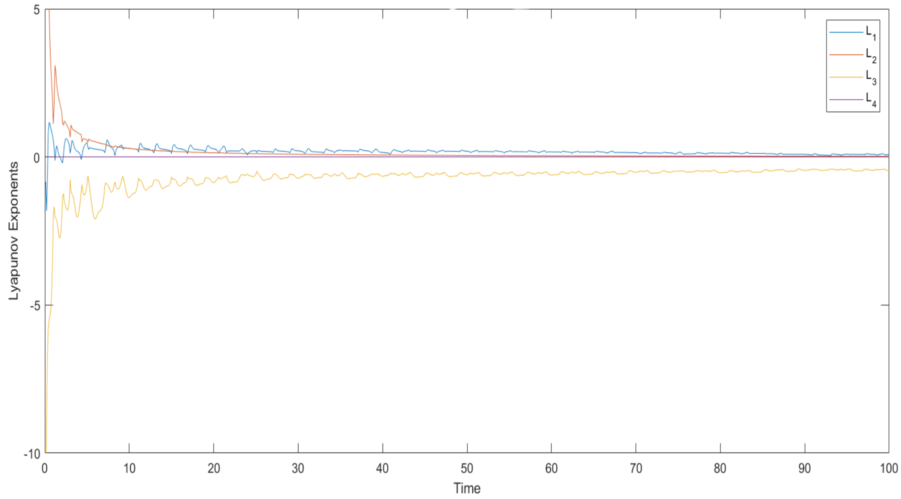


Figure 3: The Lyapunov exponents of system (3.4).

*3.2. SIR system synchronization applying a variety of control techniques*

*3.2.1. Active control*

The controllable chaotic slave system is:

$$\begin{cases} \dot{y_1} &= \alpha - \phi_0(1+\beta\sin(y_4))y_1y_2 - \lambda y_1 + u_1 \\ \dot{y_2} &= \phi_0(1+\beta\sin(y_4))y_1y_2 - (\delta+\lambda)y_2 + u_2 \\ \dot{y_3} &= \delta y_2 - \lambda y_3 + u_3 \\ \dot{y_4} &= \frac{2\pi}{T} + u_4. \end{cases} \tag{3.5}$$

where $u_i$ denotes the active control functions that will be found subsequently for $i = 1...4$. The error synchronization between systems (3.4) and (3.5).

$$\begin{cases} \dot{e_1} &= -\lambda e_1 - \phi_0(1+\beta\sin(y_4))y_1y_2 - (1+\beta\sin(x_4))x_1x_2) + u_1 \\ \dot{e_2} &= (\delta+\lambda)e_2 + \phi_0(1+\beta\sin(y_4))y_1y_2 - (1+\beta\sin(x_4))x_1x_2) + u_2 \\ \dot{e_3} &= \delta e_2 - \lambda e_3 + u_3 \\ \dot{e_4} &= u_4. \end{cases} \tag{3.6}$$

The active control laws for each $u_i$, where $i = 1, 2, 3, 4$, are formulated according to the design presented in [30], ensuring that error dynamics are global asymptotic stable. Define the control laws as follows:

$$\begin{cases} u_1 &= \phi_0(1+\beta\sin(y_4))y_1y_2 - \phi_0(1+\beta\sin(x_4))x_1x_2) + v_1 \\ u_2 &= -\phi_0(1+\beta\sin(y_4))y_1y_2 - \phi_0(1+\beta\sin(x_4))x_1x_2) + v_2 \\ u_3 &= -\delta e_2 + v_3 \\ u_4 &= -e_4 + v_4. \end{cases} \tag{3.7}$$

The error system (3.6) become:

$$\begin{cases} \dot{e_1} &= -\lambda e_1 + v_1 \\ \dot{e_2} &= -(\delta+\lambda)e_2 + v_2 \\ \dot{e_3} &= -\lambda e_3 + v_3 \\ \dot{e_4} &= -e_4 + v_4. \end{cases} \tag{3.8}$$

The system's matrix form is:

$$[\dot{e_1}\ \dot{e_2}\ \dot{e_3}\ \dot{e_4}]^T = A[e_1\ e_2\ e_3\ e_4]^T \tag{3.9}$$

and

$$[v_1\ v_2\ v_3\ v_4]^T = B[e_1\ e_2\ e_3\ e_4]^T \tag{3.10}$$

$A$ constant matrix, $B$ is chosen in the active control approach to regulate the error's dynamics.

$$B = \begin{pmatrix} a_{11} & b_{12} & c_{13} & d_{14} \\ a_{21} & b_{22} & c_{23} & d_{24} \\ a_{31} & b_{32} & c_{33} & d_{34} \\ a_{41} & b_{42} & c_{43} & d_{44} \end{pmatrix} \tag{3.11}$$

So

$$\begin{pmatrix} \dot{e_1} \\ \dot{e_2} \\ \dot{e_3} \\ \dot{e_4} \end{pmatrix} = \begin{pmatrix} -\lambda - a_{11} & b_{12} & c_{13} & d_{14} \\ a_{21} & -(\delta+\lambda) - b_{22} & c_{23} & d_{24} \\ a_{31} & b_{32} & -\lambda - c_{33} & d_{34} \\ a_{41} & b_{42} & c_{43} & -1 - d_{44} \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{pmatrix} \tag{3.12}$$

Several choices of the matrix $B$ can fulfill the error dynamics. Still, One might choose to employ matrix $B$ that satisfies the Routh-Hurwitz criterion for ensuring stability of the error system near zero. Therefore,

$$B = \begin{pmatrix} -0.99 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -0.99 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{3.13}$$

As a result, systems (3.4) and (3.5) are asymptotically synchronized, since the system previously mentioned (3.12) is asymptotically stable.

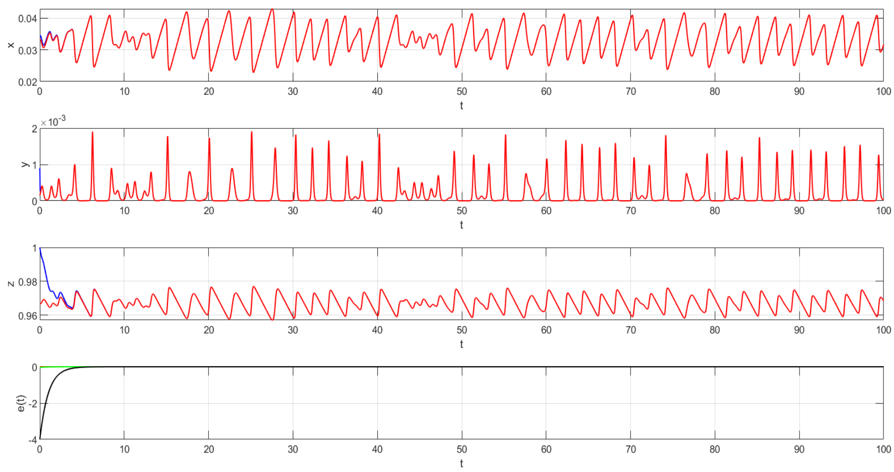


Figure 4: Synchronization of system (3.4) and (3.5) using active control .

*3.2.2. Sliding mode control*

Between systems (3.4) and (3.5), the synchronization error is:

$$\begin{cases} \dot{e_1} &= -0.01e_1 - 1510(1+\beta\sin(y_4))y_1y_2 - (1+\beta\sin(x_4))x_1x_2) + u_1 \\ \dot{e_2} &= -50.01e_2 + 1510(1+\beta\sin(y_4))y_1y_2 - (1+\beta\sin(x_4))x_1x_2) + u_2 \\ \dot{e_3} &= 50e_2 - 0.01e_3 + u_3 \\ \dot{e_4} &= u_4. \end{cases} \tag{3.14}$$

When two systems are synchronized, the problem becomes one of error control. This section will provide a sliding mode control solution to this issue. The SMC approach requires two phases to bring any systems that are chaotic into synchronization. The initial stage involves selecting appropriate sliding surfaces, that guarantee the related sliding mode error dynamics' stability and cause the synchronization error (3.14) to converge to zero. Assume the following for the sliding surfaces:

$$\begin{cases} s_{sm1} &= e_1 + \int_0^t q_1 e_1 d\tau \\ s_{sm2} &= e_2 + \int_0^t q_2 e_2 d\tau \\ s_{sm3} &= e_3 + \int_0^t q_3 e_3 d\tau \\ s_{sm4} &= e_4 + \int_0^t q_4 e_4 d\tau \end{cases} \tag{3.15}$$

In which the variables $q_i, i = 1, 2, 3, 4$ are all positive. In sliding mode, the system meets (3.16) condition.

$$\begin{cases} \dot{s_{sm1}} &= 0 \\ \dot{s_{sm2}} &= 0 \\ \dot{s_{sm3}} &= 0 \\ \dot{s_{sm4}} &= 0 \end{cases} \tag{3.16}$$

Comparable sliding mode error behaviors are represented in the following way:

$$\begin{cases} \dot{e_1} &= -q_1 e_1 \\ \dot{e_2} &= -q_2 e_2 \\ \dot{e_3} &= -q_3 e_3 \\ \dot{e_4} &= -q_4 e_4 \end{cases} \tag{3.17}$$

Let the following Lyapunov function be:

$$W(e) = \frac{1}{2}(e_1^2 + e_2^2 + e_3^2 + e_4^2) \tag{3.18}$$

Which implies that

$$\dot{W} = e_1\dot{e_1} + e_2\dot{e_2} + e_3\dot{e_3} + e_4\dot{e_4} \tag{3.19}$$

Substituting (3.17) into (3.19), we find

$$\dot{W} = -q_1 e_1^2 - q_2 e_2^2 - q_3 e_3^2 - q_4 e_4^2 \tag{3.20}$$

Since $q_1, q_2, q_3, q_4$ are positive constants, $\dot{W}$ is negative defined.The Lyapunov theory's stability principle guarantees the convergence of error dynamics (3.17) by stating that sliding motion on the sliding surface is stable. Creating a sliding mode control is the second step in ensuring that error routes end up at the sliding surface $s_{smi} = 0$, which establishes an appropriate sliding surface (3.15). The control law proposed below is meant to guarantee that sliding motion takes place:

$$\begin{cases} u_1 &= 1510[(1+\beta\sin(y_4))y_1y_2 - (1+\beta\sin(x_4))x_1x_2] + (0.01 - q_1)e_1 \\ &\quad -k.sign(s_{sm1}) \\ u_2 &= -1510[(1+\beta\sin(y_4))y_1y_2 - (1+\beta\sin(x_4))x_1x_2] + (50.01 - q_2)e_2 \\ &\quad -k.sign(s_{sm2}) \\ u_3 &= -50e_2 + (0.01 - q_3)e_3 - k.sign(s_{sm3}) \\ u_4 &= -q_4e_4 - k.sign(s_{sm4}) \end{cases} \tag{3.21}$$

where $k$ is a positive constant that needs to be determined carefully. The paths of systems (3.4) and (3.5) approach the sliding surface $s_{sm}(t) = 0$ according to the controlling rules in equation (3.21). Let be the ensuing Lyapunov function:

$$W(e) = \frac{1}{2}(s_{sm1}^2 + s_{sm2}^2 + s_{sm3}^2 + s_{sm4}^2) \tag{3.22}$$

Which implies that

$$\dot{W} = s_{sm1}\dot{s_{sm1}} + s_{sm2}\dot{s_{sm2}} + s_{sm3}\dot{s_{sm3}} + s_{sm4}\dot{s_{sm4}} \tag{3.23}$$

$$\dot{W} = s_{sm1}(\dot{e_1} + q_1 e_1) + s_{sm2}(\dot{e_2} + q_2 e_2) + s_{sm3}(\dot{e_3} + q_3 e_3) + s_{sm4}(\dot{e_4} + q_4 e_4) \tag{3.24}$$

So

$$\dot{W} \leq -k|s_{sm1}| - k|s_{sm2}| - k|s_{sm3}| - k|s_{sm4}| \tag{3.25}$$

One may demonstrate that $\dot{W}(s_{sm}) < 0$ for $s_{smi}(t) \neq 0$ by choosing $k > 0$. According to Lyapunov theory's stability principle, $s_{sm}(t)$ always converges to the sliding surface $s_{sm}(t) = 0$. Consequently, the error behaviors are asymptotically stable and converge to zero.

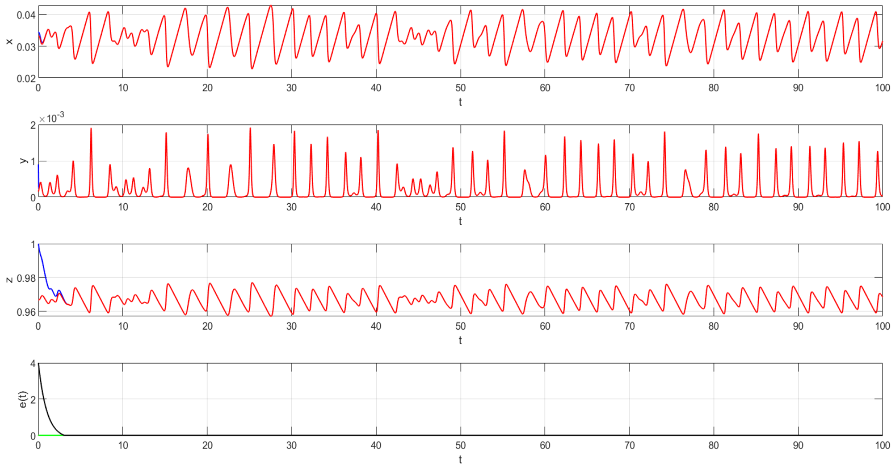

Figure 5: Synchronization of system (3.4) and (3.5) using sliding mode control.

*3.2.3. Active backstepping control*

In system (3.4) we take $\alpha = 0.01, \lambda = 0.01, \delta = 50, \phi = 1510, T = 1$. Performing the variable permutation:

$$\begin{cases} x_1 = x_4 \\ x_2 = x_3 \\ x_3 = x_2 \\ x_4 = x_1 \end{cases} \Rightarrow \begin{cases} \dot{x_1} = \dot{x_4} \\ \dot{x_2} = \dot{x_3} \\ \dot{x_3} = \dot{x_3} \\ \dot{x_4} = \dot{x_1} \end{cases} \tag{3.26}$$

Therefore, system (3.4) can be identified as:

$$\begin{cases} \dot{x_1} &= 2\pi \\ \dot{x_2} &= -0.01x_2 + 50x_3 \\ \dot{x_3} &= -50.01x_3 + 1510(1 + \beta\sin(x_1))x_3x_4 \\ \dot{x_4} &= 0.01 - 0.01x_4 - 1510(1 + \beta\sin(x_1))x_3x_4. \end{cases} \tag{3.27}$$

So system (3.5) becomes:

$$\begin{cases} \dot{y_1} &= 2\pi + u_1 \\ \dot{y_2} &= -0.01y_2 + 50y_3 + u_2 \\ \dot{y_3} &= -50.01y_3 + 1510(1 + \beta\sin(y_1))y_3y_4 + u_3 \\ \dot{y_4} &= 0.01 - 0.01y_4 - 1510(1 + \beta\sin(y_1))y_3y_4 + u_4. \end{cases} \tag{3.28}$$

In this case, the feedback control instructions $u_1, u_2, u_3, u_4$ need to be found by backstepping. The error system is:

$$\begin{cases} \dot{e_1} &= u_1 \\ \dot{e_2} &= -0.01e_2 + 50e_3 + u_2 \\ \dot{e_3} &= -50.01e_3 + 1510((1+\beta\sin(y_1))y_3y_4 - (1+\beta\sin(x_1))x_3x_4) + u_3 \\ \dot{e_4} &= -0.01e_4 - 1510((1+\beta\sin(y_1))y_3y_4 - (1+\beta\sin(x_1))x_3x_4) + u_4. \end{cases} \tag{3.29}$$

Initially, feedback control is used to restructure the system (3.29), making it conducive for the implementation of backstepping control. Let's begin by examining the feedback control law:

$$\begin{cases} u_1 &= -e_1 - e_2 \\ u_2 &= -0.99e_2 \\ u_3 &= 0.01e_3 - 1510((1+\varepsilon\sin(y_1))y_3y_4 - (1+\varepsilon\sin(x_1))x_3x_4) + e_4 \\ u_4 &= -0.99e_4 + 1510((1+\varepsilon\sin(y_1))y_3y_4 - (1+\varepsilon\sin(x_1))x_3x_4) + v. \end{cases} \tag{3.30}$$

Here, $v$ stands for a command to be determined for feedback control. With (3.30) substituted for (3.29), the form that follows may be used to receive the updated system:

$$\begin{cases} \dot{e_1} &= -e_1 - e_2 \\ \dot{e_2} &= -e_2 + 50e_3 \\ \dot{e_3} &= -50e_3 + e_4 \\ \dot{e_4} &= -e_4 + v. \end{cases} \tag{3.31}$$

We initiate the process with the Lyapunov function:

$$W_1(z_1) = \frac{1}{2}z_1^2 \tag{3.32}$$

where $z_1 = e_1$, we obtain

$$\dot{W}_1(z_1) = z_1\dot{z}_1 = z_1(-e_1 - e_2) = -z_1^2 - z_1e_2 \tag{3.33}$$

Next, we define

$$z_2 = -e_2 \tag{3.34}$$

Using (3.34), we can express (3.33) as

$$\dot{W}_1(z_1) = -z_1^2 + z_1z_2 \tag{3.35}$$

We define the second Lyapunov function:

$$W_2(z_1, z_2) = W_1(z_1) + \frac{1}{2}z_2^2 \tag{3.36}$$

We obtain

$$\dot{W}_2(z_1, z_2) = -z_1^2 + z_1z_2 + z_2\dot{z}_2 = -z_1^2 - z_2^2 + w_2(w_1 + w_2 + \dot{w}_2) \tag{3.37}$$

We define

$$z_3 = z_1 + z_2 + \dot{z}_2 = e_1 - e_2 - (-e_2 + 50e_3) = e_1 - 50e_3 \tag{3.38}$$

Thus, (3.37) becomes

$$\dot{W}_2(z_1, z_2) = -z_1^2 - z_2^2 + z_2z_3 \tag{3.39}$$

We define the third Lyapunov function:

$$W_3(z_1, z_2, z_3) = W_2(z_1, z_2) + \frac{1}{2}z_3^2 \tag{3.40}$$

We get

$$\dot{W}_3(z_1, z_2, z_3) = -z_1^2 - z_2^2 + z_2z_3 + z_3\dot{z}_3 = -z_1^2 - z_2^2 - z_3^2 + z_3(z_2 + z_3 + \dot{z}_3) \tag{3.41}$$

We define

$$\begin{aligned} z_4 &= z_2 + z_3 + \dot{z}_3 = -e_2 + e_1 - 50e_3 + \dot{e}_1 - 50\dot{e}_3 = -e_2 + e_1 - 50e_3 - e_1 - e_2 \\ &+2500e_3 - 50e_4 = -2e_2 + 2450e_3 - 50e_4 \end{aligned} \tag{3.42}$$

Hence, (3.41) becomes

$$\dot{W}_3(z_1, z_2, z_3) = -z_1^2 - z_2^2 - z_3^2 + z_3 z_4 \tag{3.43}$$

We define the fourth Lyapunov function:

$$W_4(z_1, z_2, z_3, z_4) = W_3(z_1, z_2, z_3) + \frac{1}{2} z_4^2 \tag{3.44}$$

We obtain

$$\dot{W}_4(z_1, z_2, z_3, z_4) = -z_1^2 - z_2^2 - z_3^2 - z_4^2 + z_4(z_3 + z_4 + \dot{z}_4) \tag{3.45}$$

We set

$$\begin{aligned} Z &= z_3 + z_4 + \dot{z}_4 = e_1 - 50e_3 - 2e_2 + 2450e_3 - 50e_4 - 2\dot{e_2} + 2450\dot{e_3} - 50\dot{e_4} \\ &= e_1 - 50e_3 - 2e_2 + 2450e_3 - 50e_4 - 2(-e_2 + 50e_3) + 2450(-50e_3 + e_4) \\ &-50(-e_4 + v) = e_1 - 120200e_3 + 2450e_4 - 50v \end{aligned} \tag{3.46}$$

We define $v$ as:

$$v = 0.02e_1 - 2404e_3 + 49e_4 + Qz_4 \tag{3.47}$$

when a positive constant $Q > 0$ is present. We obtain $Z = -50Qz_4$ by substituting (3.47) into (3.46). Consequently, we can simplify equation (3.45) as follows:

$$\dot{W}_4(z_1, z_2, z_3, z_4) = -z_1^2 - z_2^2 - z_3^2 - (1 + 50Q)z_4^2 \tag{3.48}$$

This quadratic equation is negatively definite. Therefore, the Lyapunov theory's stability principle ensures global and exponential stability of the system (3.31). The control laws in the active backstepping method are then defined as:

$$\begin{cases} u_1 &= -e_1 - e_2 \\ u_2 &= -0.99e_2 \\ u_3 &= 0.01e_3 - 1510((1 + \beta \sin(y_1))y_3 y_4 - (1 + \beta \sin(x_1))x_3 x_4) + e_4 \\ u_4 &= -0.99e_4 + 1510((1 + \beta \sin(y_1))y_3 y_4 - (1 + \beta \sin(x_1))x_3 x_4) + 0.02e_1 \\ & -2404e_3 + 49e_4 - 50Qz_4. \end{cases} \tag{3.49}$$

with $z_4 = -2e_2 + 2450e_3 - 50e_4$,

*3.2.4. Adaptative control*

Using unknown parameters, we rewrite systems (3.4) and (3.5) in the way that follows:

$$\begin{cases} \dot{x} &= h(x) + H(x)\xi \\ \dot{y} &= h(y) + H(y)\xi + U. \end{cases} \tag{3.50}$$

where

$$\dot{x} = \begin{pmatrix} \dot{x_1} & \dot{x_2} & \dot{x_3} & \dot{x_4} \end{pmatrix}^T \tag{3.51}$$

$$\dot{y} = \begin{pmatrix} \dot{y_1} & \dot{y_2} & \dot{y_3} & \dot{y_4} \end{pmatrix}^T \tag{3.52}$$

and

$$h(x) = h(y) = \begin{pmatrix} 0 & 0 & 0 & 2\pi \end{pmatrix}^T \tag{3.53}$$

$$H(x) = \begin{pmatrix} 1 & -x_1 & -(1 + \beta \sin(x_4))x_1 x_2 & -\phi \sin(x_4)x_1 x_2 & 0 \\ 0 & -x_2 & (1 + \beta \sin(x_4))x_1 x_2 & \phi \sin(x_4)x_1 x_2 & -x_2 \\ 0 & -x_3 & 0 & 0 & x_2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{3.54}$$

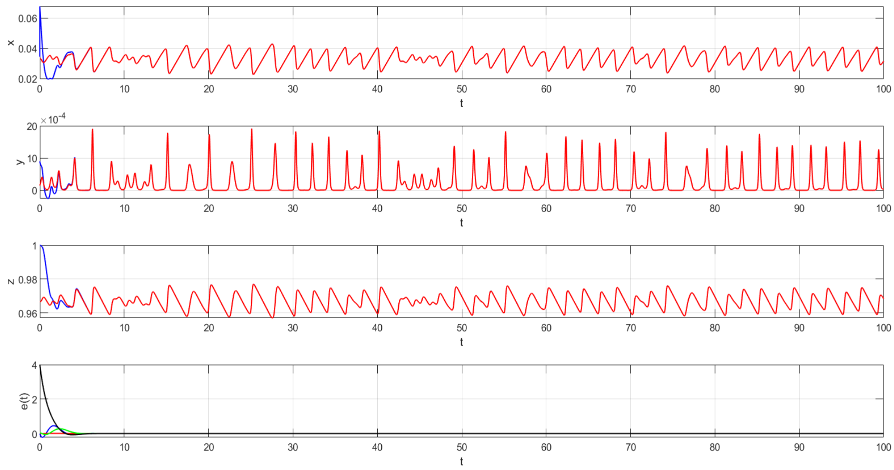

Figure 6: Synchronization of system (3.4) and (3.5) using active backstepping control.

$$H(y)=\begin{pmatrix} 1 & -y_1 & -(1+\beta\sin(y_4))y_1y_2 & -\phi\sin(y_4)y_1y_2 & 0\\ 0 & -y_2 & (1+\beta\sin(y_4))y_1y_2 & \phi\sin(y_4)y_1y_2 & -y_2\\ 0 & -y_3 & 0 & 0 & y_2\\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{3.55}$$

$$\xi=\begin{pmatrix}\alpha & \lambda & \phi & \beta & \delta\end{pmatrix}^T; U=\begin{pmatrix}u_1 & u_2 & u_3 & u_4\end{pmatrix}^T \tag{3.56}$$

The error in complete synchronization between systems (3.4) and (3.5) is given by:

$$\dot{e}_i=(H(y)-H(x))\xi+U \tag{3.57}$$

We define adaptative control as:

$$U=(H(x)-H(y))\tilde{\xi}-ke \tag{3.58}$$

where

$$\tilde{\xi}=\begin{pmatrix}\tilde{\alpha} & \tilde{\lambda} & \tilde{\phi} & \tilde{\beta} & \tilde{\delta}\end{pmatrix}^T, e=\begin{pmatrix}e_1 & e_2 & e_3 & e_4.\end{pmatrix}^T \tag{3.59}$$

and $k=diag(k_i)\in\mathbb{R}^{4\times4}, k_i>0$ .So

$$\begin{cases} u_1 &= \tilde{\lambda}e_1+\tilde{\phi}((1+\beta\sin(y_4))y_1y_2-(1+\beta\sin(x_4))x_1x_2)+\phi\tilde{\beta}(\sin(y_4)y_1y_2\\ & -\sin(x_4)x_1x_2)-k_1e_1\\ u_2 &= \tilde{\lambda}e_2-\tilde{\phi}((1+\beta\sin(y_4))y_1y_2-(1+\beta\sin(x_4))x_1x_2)-\phi\tilde{\beta}(\sin(y_4)y_1y_2\\ & -\sin(x_4)x_1x_2)+\tilde{\delta}e_2-k_2e_2\\ u_3 &= \tilde{\lambda}e_3-\tilde{\delta}e_2-k_3e_3\\ u_4 &= -k_4e_4 \end{cases} \tag{3.60}$$

According to the parameter estimation adjustment laws below:

$$\dot{\tilde{\xi}}=[H(y)-H(x)]^Te+\eta(\xi-\tilde{\xi}) \tag{3.61}$$

hence

$$\begin{cases} \dot{\tilde{\alpha}} &= \eta(\alpha-\tilde{\alpha})\\ \dot{\tilde{\lambda}} &= -(e_1^2+e_2^2+e_3^2)+\eta(\lambda-\tilde{\lambda})\\ \dot{\tilde{\phi}} &= (e_2-e_1)((1+\beta\sin(y_4))y_1y_2(1+\beta\sin(x_4))x_1x_2)+\eta(\phi-\tilde{\phi})\\ \dot{\tilde{\beta}} &= \phi(e_2-e_1)(\sin(y_4)y_1y_2-\sin(x_4)x_1x_2)+\eta(\beta-\tilde{\beta})\\ \dot{\tilde{\delta}} &= -e_2^2+e_2e_3+\eta(\delta-\tilde{\delta}) \end{cases} \tag{3.62}$$

where

$$\eta = diag(\eta_i), \eta_i > 0, i = 1...5,$$

Let's define Lyapunov's function as follows:

$$W(e) = e_1^2 + e_2^2 + e_3^2 + e_4^2 + e_\alpha^2 + e_\lambda^2 + e_\phi^2 + e_\beta^2 + e_\delta^2 \tag{3.63}$$

With the choice of parameters (3.62). $\dot{W}$ is a function with a negative definite. Consequently, the error dynamics (3.57) show global asymptotic stability according to Lyapunov stability theory.

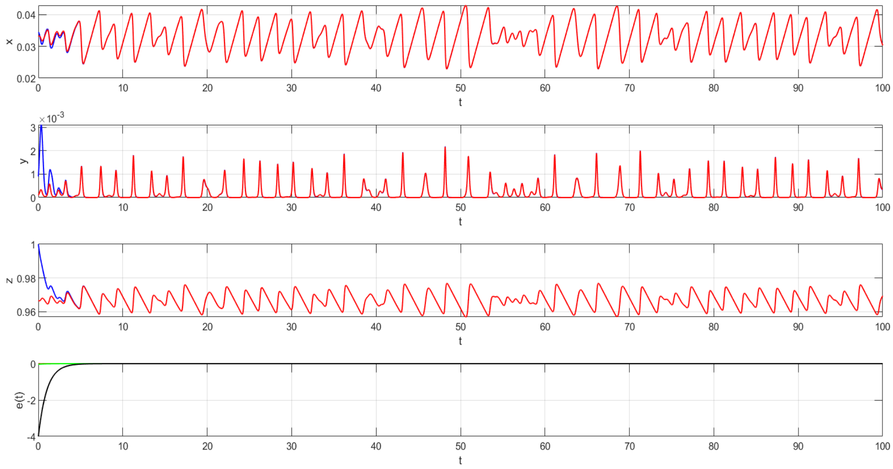

Figure 7: Synchronization of system (3.4) and (3.5) using adaptative control.

### *3.3. Comparing synchronization control methods*

Based on the work of [18]. To synchronize the forced SIR system, the efficiency of the AC, SMC, ABC, and AdC techniques are compared. For comparison, the necessary control energy $UE$ and the integral square error $IE$ are taken into account. This is because the relationship between error dynamics and control input can vary greatly depending on the specific system being analyzed. Therefore, it is important to carefully consider all factors before determining the most effective control strategy. Consequently, $I = g(IE, UE)$ defines an appropriate performance index as a function of $IE$ and $UE$. which have the following definitions: $IE = \int_0^\infty e^2(t)dt$ , $UE = \int_{-\infty}^{+\infty} |u|^2 dt$, where $e$ is the error vecteur and $u$ is the input for control. Three distinct efficiency indicator samples are given for comparison. These are:

**Case 1**: When $IE$ and $UE$ have identical weight.

$$I = g(IE, UE) = \sum_{k=1}^{n} (\frac{1}{2} IE_k + \frac{1}{2} UE_k). \tag{3.64}$$

**Case 2**: If $IE$ and $UE$ have weight of 40% and 60%, respectively.

$$I = g(IE, UE) = \sum_{k=1}^{n} (\frac{2}{5} IE_k + \frac{3}{5} UE_k). \tag{3.65}$$

**Case 3**: When $IE$ has a 60% weight and $UE$ has a 40% weight.

$$I = g(IE, UE) = \sum_{k=1}^{n} (\frac{3}{5} IE_k + \frac{2}{5} UE_k) \tag{3.66}$$

Table 1: Comparaison of performance index for the fourth control techniques first case

| Initial conditions | | | | AC | SMC | ABC | AdC |
|---|---|---|---|---|---|---|---|
| 0.0345 | 0.0002 | 1 | 5 | **4** | 8.05 | 15.75 | 8 |
| 0.0445 | 0.0102 | 1.01 | 5.01 | **4.02** | 8.17 | $1.2 \times 10^3$ | 8.66 |
| 0.0645 | 0.0302 | 1.03 | 5.03 | **4.09** | 8.72 | $8.9 \times 10^3$ | 12.82 |
| 0.0845 | 0.0502 | 1.05 | 5.05 | **4.26** | 10.25 | $2.5 \times 10^4$ | 19.84 |
| 0.1045 | 0.0702 | 1.07 | 5.07 | **4.62** | 14 | $5.7 \times 10^4$ | 32.26 |
| 0.1245 | 0.0902 | 1.09 | 5.09 | **5.31** | 22.3 | $1.2 \times 10^5$ | 54.76 |
| 0.1445 | 0.1102 | 1.11 | 5.11 | **6.50** | 39.69 | $2.2 \times 10^5$ | 92.1 |
| 0.1645 | 0.1302 | 1.13 | 5.13 | **8.4** | 75.16 | $3.9 \times 10^5$ | 194.25 |
| 0.1845 | 0.1502 | 1.15 | 5.15 | **11.24** | 142.12 | $6.5 \times 10^5$ | 375.82 |
| 0.2045 | 0.1702 | 1.17 | 5.17 | **15.32** | 304.79 | $10^6$ | 545.6 |

Table 2: Comparaison of performance index for the fourth control techniques second case

| Initial conditions | | | | AC | SMC | ABC | AdC |
|---|---|---|---|---|---|---|---|
| 0.0345 | 0.0002 | 1 | 5 | **3.20** | 8.07 | 17.31 | 8 |
| 0.0445 | 0.0102 | 1.01 | 5.01 | **3.22** | 8.20 | $1.4 \times 10^3$ | 8.79 |
| 0.0645 | 0.0302 | 1.03 | 5.03 | **3.29** | 8.85 | $1.1 \times 10^4$ | 13.76 |
| 0.0845 | 0.0502 | 1.05 | 5.05 | **3.47** | 10.67 | $3 \times 10^4$ | 22.16 |
| 0.1045 | 0.0702 | 1.07 | 5.07 | **3.89** | 15.15 | $6.8 \times 10^4$ | 37.05 |
| 0.1245 | 0.0902 | 1.09 | 5.09 | **4.70** | 25.1 | $1.4 \times 10^5$ | 64.03 |
| 0.1445 | 0.1102 | 1.11 | 5.11 | **6.11** | 45.94 | $2.6 \times 10^5$ | 108.82 |
| 0.1645 | 0.1302 | 1.13 | 5.13 | **8.37** | 88.44 | $4.6 \times 10^5$ | 231.38 |
| 0.1845 | 0.1502 | 1.15 | 5.15 | **11.76** | 176.03 | $7.8 \times 10^5$ | 427.66 |
| 0.2045 | 0.1702 | 1.17 | 5.17 | **16.64** | 364 | $1.2 \times 10^6$ | 652.98 |

Table 3: Comparaison of performance index for the fourth control techniques third case

| Initial conditions | | | | AC | SMC | ABC | AdC |
|---|---|---|---|---|---|---|---|
| 0.0345 | 0.0002 | 1 | 5 | **4.80** | 8.03 | 14.18 | 8 |
| 0.0445 | 0.0102 | 1.01 | 5.01 | **4.83** | 8.13 | 928.48 | 8.54 |
| 0.0645 | 0.0302 | 1.03 | 5.03 | **4.89** | 8.59 | 7049.7 | 11.88 |
| 0.0845 | 0.0502 | 1.05 | 5.05 | **5** | 9.83 | $2 \times 10^4$ | 17.51 |
| 0.1045 | 0.0702 | 1.07 | 5.07 | **5.22** | 12.85 | $4.5 \times 10^4$ | 27.47 |
| 0.1245 | 0.0902 | 1.09 | 5.09 | **5.66** | 19.51 | $9.2 \times 10^4$ | 45.48 |
| 0.1445 | 0.1102 | 1.11 | 5.11 | **6.89** | 33.43 | $1.7 \times 10^5$ | 75.37 |
| 0.1645 | 0.1302 | 1.13 | 5.13 | **8.43** | 61.83 | $3.1 \times 10^5$ | 157.11 |
| 0.1845 | 0.1502 | 1.15 | 5.15 | **10.72** | 120.21 | $5.2 \times 10^5$ | 287.99 |
| 0.2045 | 0.1702 | 1.17 | 5.17 | **14** | 245.56 | $8.3 \times 10^5$ | 438.23 |

The index below $k$ is the summation index ranging from 1 to 4, Thus, for $k = 1, \ldots, 4$, $(IE_k)$ and $(UE_k)$ are calculated.

Chaotic systems are characterized by their sensitivity to initial conditions. we take ten different conditions, and we notice that the synchronization is complete and also according to the table the best control for the system whatever the index is the active control

## 4. Conclusion

Using well-known control strategies like Active Control (AC), Sliding Mode Control (SMC), Active Backstepping (ABC), and Adaptive Control (AdC) in conjunction with numerical simulations, SIR system synchronization was accomplished in this work. The control line for this system, which is the active control that frequently permits precise and fine-tuning system parameters and might be essential for attaining precise synchronization across systems, can be found by employing a well-defined performance index, can enable quicker reaction times, which is extremely important for our system since synchronization must be achieved soon further restate that the SMC comes in second, followed by AdC. One promising direction for future research is to explore the effectiveness of hybrid control strategies on synchronization dynamics in epidemiological systems. For example, combining Active Control with Sliding Mode Control or Adaptive Control could leverage the strengths of each approach, potentially leading to even more precise and robust synchronization. Additionally, studying the impact of these hybrid strategies in various scenarios, such as different population sizes or varying transmission rates, could provide deeper insights into their practical applications. Another area worth investigating is the integration of real-time data from ongoing epidemics to dynamically adjust the control parameters, thereby enhancing the responsiveness and accuracy of the synchronization process. Finally, examining the interactions between epidemiological systems and other complex networks, such as transportation or social networks, could offer valuable perspectives on managing disease spread more effectively.

## References

[1] M. M. Al-Sawalha and M. S. Md Noorani, *Anti-synchronization of two hyperchaotic systems via nonlinear control*, Communications in Nonlinear Science and Numerical Simulation **14:8** (2009), 3402–3411.

[2] P. G. Barrientos, J. Á. Rodríguez, and A. Ruiz-Herrera, *Chaotic dynamics in the seasonally forced SIR epidemic model*, Journal of Mathematical Biology **75** (2017), 1655–1668.

[3] G. Chen and X. Dong, *From chaos to order: methodologies, perspectives, and applications*, World Scientific, 5 Toh Tuck Link, Singapore 596224 (1998).

[4] X. Chen, J. H. Park, J. Cao, and J. Qiu, *Sliding mode synchronization of multiple chaotic systems with uncertainties and disturbances*, Applied Mathematics and Computation **308** (2017), 161–173.

[5] I. Cooper, A. Mondal, and C. G. Antonopoulos, *A SIR model assumption for the spread of COVID-19 in different communities*, Chaos, Solitons & Fractals **139** (2020), 110057.

[6] J. P. S. M. de Carvalho and A. A. Rodrigues, *Strange attractors in a dynamical system inspired by a seasonally forced SIR model*, Physica D: Nonlinear Phenomena **434** (2022), 133268.

[7] K. Dietz, *The incidence of infectious diseases under the influence of seasonal fluctuations*, Mathematical Models in Medicine: Workshop, Mainz, March (1976), 1–15.

[8] J. Duarte, C. Januário, N. Martins, S. Rogovchenko, and Y. Rogovchenko, *Chaos analysis and explicit series solutions to the seasonally forced SIR epidemic model*, Journal of Mathematical Biology **78** (2019), 2235–2258.

[9] E. M. Elabbasy, H. N. Agiza, and M. M. El-Dessoky, *Adaptive synchronization of a hyperchaotic system with uncertain parameter*, Chaos, Solitons & Fractals **30:5** (2006), 1133–1142.

[10] E. Fisher, *A biological approach for financial network contagion based on the Susceptible-Infected-Recovered (SIR) model*, Análisis Económico **28:69** (2013), 109–128.

[11] R. M. Hillary and M. A. Bees, *Synchrony & chaos in patchy ecosystems*, Bulletin of Mathematical Biology **66** (2004), 1909–1931.

[12] Z. Xiaoshan , L. Zhenbo and L. Shuang, *Synchronization of a chaotic finance system*, Applied mathematics and Computation **217**(2011), 6031–6039.

[13] W. O. Kermack and A. G. McKendrick, *A contribution to the mathematical theory of epidemics*, Proceedings of the Royal Society of London. Series A, Containing Papers of a Mathematical and Physical Character **115:772** (1927), 700–721.

[14] Y. Lei, W. Xu, J. Shen, and T. Fang, *Global synchronization of two parametrically excited systems using active control*, Chaos, Solitons & Fractals **28:2** (2006), 428–436.

[15] M. E. J. Newman, *Spread of epidemic disease on networks*, Physical Review E **66:1** (2002), 016128.

[16] L. M. Pecora and T. L. Carroll, *Synchronization in chaotic systems*, Physical Review Letters **64:8** (1990), 821.

[17] S. S. Sharbayta, B. Buonomo, A. d'Onofrio, and T. Abdi, *'Period doubling' induced by optimal control in a behavioral SIR epidemic model*, Chaos, Solitons & Fractals **161** (2022), 112347.

[18] P. P. Singh and B. K. Roy, *Comparative performances of synchronisation between different classes of chaotic systems using three control techniques*, Annual Reviews in Control **45** (2018), 152–165.

[19] A. Tarammim and M. T. Akter, *A comparative study of synchronization methods of Rucklidge chaotic systems with design of active control and backstepping methods*, International Journal of Modern Nonlinear Theory and Application **11:2** (2022), 31–51.

[20] Y. Ucakan, S. Gulen, and K. Koklu, *Analysing of tuberculosis in Turkey through SIR, SEIR and BSEIR mathematical models*, Mathematical and Computer Modelling of Dynamical Systems **27:1** (2021), 179–202.

[21] S. Vaidyanathan, *Anti-synchronization of four-scroll chaotic systems via sliding mode control*, International Journal of Information Technology, Control, and Automation **1:1** (2011), 1–15.

[22] S. Vaidyanathan and S. Sampath, *Anti-synchronization of four-wing chaotic systems via sliding mode control*, International Journal of Automation and Computing **9** (2012), 274–279.

[23] Z. Wu and X. Fu, *Combination synchronization of three different order nonlinear systems using active backstepping design*, Nonlinear Dynamics **73** (2013), 1863–1872.

[24] A. Tarammim and M. T. Akter, *A comparative study of synchronization methods of Rucklidge chaotic systems with design of active control and backstepping methods*, International Journal of Modern Nonlinear Theory and Application **11:2** (2022), 31–51.

[25] Y. Xu, L. Wei, X. Jiang, and Z. Zhu, *Complex dynamics of a SIRS epidemic model with the influence of hospital bed number*, Discrete and Continuous Dynamical Systems-B **26:12** (2021), 6229–6252.

[26] N. E. Hamri, *Une méthode de contrôle linéaire des systèmes non linéaire*, Magreb Math. Rev **9** (2000), 105–113.

[27] W. Laouira and N. E. Hamri, *New Design of Stability Study for Linear and nonlinear Feedback Control of Chaotic Systems*, Nonlinear Dynamics and Systems Theory **22** (2022), 414–423.

[28] M. C. Benkara and N. E. Hamri, *Dynamics analysis of the coexistence equilibrium for a differential algebraic biological economic system with the hybrid functional response*, Dynamics of Continuous Discrete and Impulsive Systems **30** (2023), 375–394.

[29] J. Zu and L. Wang, *Periodic solutions for a seasonally forced SIR model with impact of media coverage*, Advances in Difference Equations (2015), 1–10.

[30] P. P. Singh, J. P. Singh, and B. K. Roy, *Synchronization and anti-synchronization of Lu and Bhalekar–Gejji chaotic systems using nonlinear active control*, Chaos, Solitons & Fractals **69** (2014), 31–39.

[31] T. Cemil, *On the stability of solutions to a certain fourth-order delay differential equation*, Nonlinear Dynamics **51** (2008), 71–81.

[32] T. Cemil, *Stability and bounded of solutions to non-autonomous delay differential equations of third order*, Nonlinear Dynamics **62** (2010), 945–953.

[33] T. Cemil, *Some stability and boundedness conditions for non-autonomous differential equations with deviating arguments*, Electronic Journal of Qualitative Theory of Differential Equations **2010** (2010), 1–12.

[34] T. Cemil, *Some remarks on the stability and boundedness of solutions of certain differential equations of fourth-order*, Computational Applied Mathematics **26** (2007), 1–17.