Available online at http://jprm.sms.edu.pk/
Journal of Prime Research in Mathematics, 20(2) (2024), 77-92

Journal of Prime Research in Mathematics

Edge Irregular Reflexive Labeling of Some Families of Ladder Graphs

Mohammed Ali Alghamdi[a], Dina Abuzaid[a], Ali Ahmad[b], Muhammad Faisal Nadeem[c,*]

[a]*Department of Mathematics, Faculty of Science, King Abdulaziz University, Jeddah 21589, Saudi Arabia.*
[b]*Department of Computer Science, College of Engineering and Computer Science, Jazan University, Jazan, Saudi Arabia.*
[c]*Department of Mathematics, COMSATS University Islamabad, Lahore Campus, 54000, Lahore, Pakistan.*

Abstract

The aim of this paper is to investigate reflexive edge strength in graph theory, defined as the specialized area of an edge that is irregularly labeled, where both vertices and edges are labeled. The reflexive edge strength, $res(G)$, is the minimal value of k for which the sum of weights of any two different edges in a graph is distinct. In this paper, reflexive edge strength of b-subdivided ladder graphs and the triangular ladder graph studied.

Keywords: Edge irregular reflexive labeling, reflexive edge strength; ladder graph, sub-divided graphs.
2010 MSC: 05C78, 05C38

1. Introduction

Graph labeling is the procedure in graph theory where either vertices or edges or both in a graph are assigned labels, usually numbers or other symbols according to certain pre-defined rules. The graph labeling can be vertex labeling, edge labeling, or total labeling and serves different purposes. Graph labeling has wide applications to computer science, telecommunications, and biology. Graph labeling in network design, for instance, enables the effective frequency or channel allocation so that interferences can be avoided. In coding theory, it could be employed to assist in error detection and correction, whereas in DNA sequencing, molecular biology labeling may represent different types of relationships or interactions among genes or proteins. Graph labeling also has some practical applications in scheduling, where tasks are represented as vertices, and dependencies are shown as edges.

In 1988 Chartand et al. [1] gave the idea for edge labeling in a special way that the weight of any two distinct vertices is not equal. This labeling is named as irregular labeling and the maximum number k that

*Corresponding author
Email addresses: mamalgamdi@kau.edu.sa (Mohammed Ali Alghamdi), dabuzaid@kau.edu.sa (Dina Abuzaid), ahmadsms@gmail.com (Ali Ahmad), mfaisalnadeem@ymail.com (Muhammad Faisal Nadeem)

Received : 16 Oct 2024*; Accepted:* 16 November 2024*; Published Online:* 20 November 2024.

is used in graph G for irregular labeling is called irregularity strength *s(G).* In 2012 Al-Mushayt et al. [2] calculates the exact value of edge irregularity strength of hexagonal grid graphs. Amar et al. calculates irregularity strength of trees [3].

Baca et al. [4] get motivations from these papers and gave a new idea of edge irregular total k-labeling of a graph (G) i.e. If *V(G)* and *E(G)* are the vertex and edge set of a graph G then $f: V \cup E \rightarrow 1, 2, 3, 4, \ldots, k$ is an edge irregular total k-labeling if the weight of any two different edges is different i.e. for all edges $w_f\left(xy\right) \neq\ w_f\left(x^{'}y^{'}\right),\ forall\ xy \neq x^{'}y^{'}$, where $w_f\left(xy\right) = f\left(x\right) + f\left(xy\right) + f(y)$. Baca et al. [5] calculates total edge irregularity strength of prism. Ivančo and Jendro calculates the value of total edge irregularity strength of tress [6].

Tanna et al. in [7] give a new thought of edge irregular reflexive k-labeling i.e. If we label both the edges and vertices of the graph such that

$$f_e : E\left(G\right) \rightarrow \{1, 2, 3, 4, \ldots, k_e\}$$

$$f_v : V\left(G\right) \rightarrow \{0, 2, 4, 6, \ldots, 2k_v\}.$$

Where $k = \max\{k_e, 2k_v\}$. Then this kind of labeling is named as edge irregular reflexive k-labeling whenever weight of $e_n \neq weight\ of\ e_m\ \forall\ e_n \neq e_m\ and\ e_n, e_m \in E(G)$, here the minimum possible value of k in which graph exists such sort of labeling is named as reflexive edge strength and written as $res(G)$.

Edge irregular reflexive labeling has been one of the most important areas of study in graph theory, whereby labeling of edges is performed so that certain specific conditions on irregularity are met. In this review, some key found studies put together various graph structures and labeling techniques.

The reflexive strength of edges can make the optimization of designs in networks, for instance, in telecommunication, where it ensures that there are unique paths or routes hence time avoiding interference or collision. Labeling edges in networks, such that paths have different values, could help in the algorithms routing to ensure at least congestion and maximum reliability. The distribution of computing systems needs a much-needed resource control and allocation at different nodes uniquely for the prevention of possible conflicts or redundancies. The reflexive edge strength can be used to label various links between distributed resources uniquely, and their usage in load balancing can help in much better processing with reduced conflicts.

Agustin et al. in [8] initiated the study of edge irregular reflexive labeling with some applications to tree graphs. Their work presented fundamental insights on how the reflexive labeling could be applied systematically to tree structures, thereby setting initial benchmarks for further extended graph analyses. Extending the investigation on corona graphs, Indriati and Rosyida [9, 10] continued consideration on the specific topic of the corona of path graphs and related graph types. Their two related publications this year provide a thorough review of how the corona operation impacts embeddability number along with other parameters of edge irregular reflexive labeling.

Further broadening the scope, Yoong et al. [11] considered the corona product of graphs with paths, addressing challenges and remedies on keeping the edge irregularity in these composite structures. In the 2021 study, subtle differences about how path graphs combine with other graph types in reflexive labeling constraints were placed in context. Budi et al. presented in [12] an investigation of the so-called tadpole graphs $T_{m,1}$ and $T_{m,2}$ examining how the Edge Irregular Reflexive Labeling could be used on such hybrid structures containing cycles and paths. This paper showed that reflexive labeling methods were flexible enough to be adapted to more complicated configurations of graphs. Junetty et al. [13], focused on palm tree graphs $C_3 - B_{2,r}$ and $C_3 - B_{3,r}$, presenting for the classes of such graphs exact methods by which edge irregular reflexive labelings are transmitted. Their results contributed to a better understanding of how some transformations of graphs did influence the approach to labeling.

For the banana tree graph $B_{2,n}$ and $B_{3,n}$, Novelia and Indriati [14] gave some properties under which it is possible to have irregular labeling. The symmetry and structure of such a graph where labeling is possible were evident through their study. Agustin et al. [15] focus mainly on almost regular graphs and their reflexive edge strength, their 2021 study shed a significant amount of light on how near-regularity in graph structures affects robustness and applicability of reflexive labeling schemes. Finally, the last related

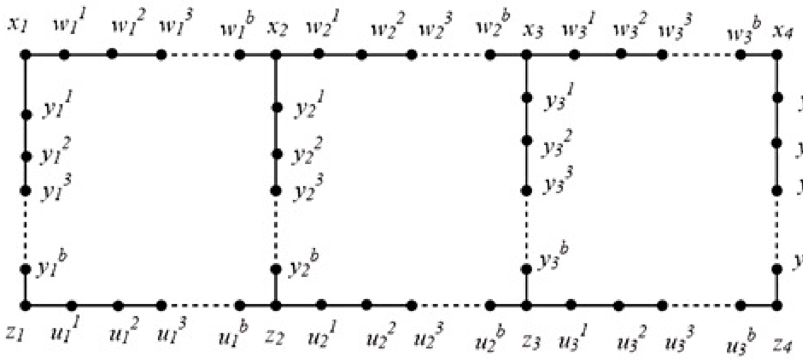

Figure 1: Subdivider ladder graph L_4^b

work which is found is by Setiawan and Indriati [16] on sun graphs and the corona of cycle graphs with null graphs containing two vertices. Their work provided insights into the complexities introduced by bringing together different graph types and resultant impact thereof on edge irregular reflexive labeling.

Let us recall the following lemma proved in [7].

Lemma 1.1. *For every graph G, $res(G) \geq \begin{cases} \left\lceil \frac{|E|}{3} \right\rceil, & |E| \not\equiv 2,3 (mod\ 6) \\ \left\lceil \frac{|E|}{3} \right\rceil + 1, & |E| \equiv 2,3 (mod\ 6) \end{cases}$*

2. Main Results

In this section, first we show the results of the ladder graph.

2.1. Reflexive edge strength of sub-divided ladder graph

Let L_a^b be the b sub-divisions of the ladder graph. It is formed by the vertex set $V\left(L_a^b\right) = \{x_i;\ 1 \leq i \leq a\} \bigcup \{z_i;\ 1 \leq i \leq a\} \bigcup \{w_i^j;\ 1 \leq i \leq a-1,\ 1 \leq j \leq b\} \bigcup \{u_i^j;\ 1 \leq i \leq a-1,\ \ 1 \leq j \leq b\} \bigcup \{y_i^j;\ 1 \leq i \leq a,\ \ 1 \leq j \leq b\}$ and edge set $E\left(L_a^b\right) = \{x_i w_i^1; 1 \leq i\ \leq a-1\} \bigcup \{w_i^j w_i^{j+1};\ 1 \leq i\ \leq a-1,\ 1 \leq j\ \leq b-1\} \bigcup \{w_i^j x_{i+1};\ 1 \leq i\ \leq a,\ \ \ j = b\} \bigcup \{x_i y_i^1; 1 \leq i\ \leq a\} \bigcup \{y_i^j y_i^{j+1};\ 1 \leq i\ \leq a,\ \ 1 \leq j\ \leq b-1\} \bigcup \{y_i^j z_i;\ 1 \leq i\ \leq a,\ \ \ j = b\} \bigcup \{z_i u_i^1; 1 \leq i\ \leq a-1\} \bigcup \{u_i^j u_i^{j+1};\ 1 \leq i\ \leq a-1,\ 1 \leq j\ \leq b-1\} \bigcup \{u_i^j z_{i+1};\ 1 \leq i\ \leq a,\ \ \ j = b\}$. Cardinality of edges in L_a^b is $3\,(ab+a) - 2(b+1)$. See Figure 1.

Theorem 2.1. *Let L_a^b be the b sub-divisions of the ladder graph, then for $a > 1$, $1 \leq b \leq 5$,*

$$res\left(L_a^b\right) = \begin{cases} \left\lceil \frac{3(ab+a)-2(b+1)}{3} \right\rceil; & |E| \not\equiv 2,3\ (mod\ 6) \\ \left\lceil \frac{3(ab+a)-2(b+1)}{3} \right\rceil + 1; & |E| \equiv 2,3\ (mod\ 6) \end{cases}$$

Proof. Let L_a^b be the b sub-divisions of the ladder graph then the lower bound of reflexive edge strength is given by Lemma 2.1 is

$$res\left(L_a^b\right) \geq \begin{cases} \left\lceil \frac{|3(ab+a)-2(b+1)|}{3} \right\rceil; & |E| \not\equiv 2,3\ (mod\ 6) \\ \left\lceil \frac{|3(ab+a)-2(b+1)|}{3} \right\rceil + 1; & |E| \equiv 2,3\ (mod\ 6) \end{cases}$$

To show that

$$res\left(L_a^b\right) \leq \begin{cases} \left\lceil \frac{|3(ab+a)-2(b+1)|}{3} \right\rceil; & |E| \not\equiv 2,3\ (mod\ 6) \\ \left\lceil \frac{|3(ab+a)-2(b+1)|}{3} \right\rceil + 1; & |E| \equiv 2,3\ (mod\ 6) \end{cases}$$

We define vertex and edge labeling in the following five different cases for $1 \leq b \leq 5$:

Case 1. When $b = 1$
We define a vertex labeling

$$f(x_1) = 0, \ f\left(x_1 y_1^1\right) = 2, f\left(y_1^1 z_1\right) = 1$$

$$f(x_{i+1}) = 2i + 2, \quad 1 \leq i \leq a - 1, \quad a = 2, 3, 4, \ldots$$

$$f\left(w_i^1\right) = 2i - 2, \quad 1 \leq i \leq a - 1, \quad a = 2, 3, 4, \ldots$$

$$f\left(y_{2i-1}^1\right) = 4i - 4, \quad 1 \leq i \leq \frac{a+1}{2}, \quad a = 1, 3, 5, \ldots$$

$$f\left(y_{2i}^1\right) = 4i - 4, \quad 1 \leq i \leq \frac{a}{2}, \quad a = 2, 4, 6, \ldots$$

$$f(z_i) = 2i, \quad 1 \leq i \leq a - 1, \quad a = 2, 3, 4, \ldots$$

$$f\left(u_i^1\right) = 2i - 2, \quad 1 \leq i \leq a - 1, \quad a = 2, 3, 4, \ldots$$

And we define an edge labeling

$$f\left(x_i w_i^1\right) = 2i - 1, \quad 1 \leq i \leq a - 1, \quad a = 2, 3, 4, \ldots$$

$$f\left(w_i^1 x_{i+1}\right) = 2i + 2, \quad 1 \leq i \leq a - 1, \quad a = 2, 3, 4, \ldots$$

$$f\left(x_{2i} y_{2i}^1\right) = 4i - 1, \quad 1 \leq i \leq \frac{a}{2}, \quad a = 2, 4, 6, \ldots$$

$$f\left(x_{2i+1} y_{2i+1}^1\right) = 4i - 1, \quad 1 \leq i \leq \frac{a-1}{2}, \quad a = 3, 5, 7, \ldots$$

$$f\left(y_{2i}^1 z_{2i}\right) = 4i - 2, \quad 1 \leq i \leq \frac{a}{2}, \quad a = 2, 4, 6, \ldots$$

$$f\left(y_{2i+1}^1 z_{2i+1}\right) = 4i - 2, \quad 1 \leq i \leq \frac{a-1}{2}, \quad a = 3, 5, 7, \ldots$$

$$f\left(z_i u_i^1\right) = 2i, \quad 1 \leq i \leq a - 1$$

$$f\left(u_i^1 z_{i+1}\right) = 2i - 1, \quad 1 \leq i \leq a - 1$$

Edge weights are shown below which can be seen that all weights are distinct

$$w_t\left(x_i w_i^1\right) = \begin{cases} 1; & i = 1 \\ 6i - 3; & i > 1 \end{cases}$$

$$w_t\left(w_i^1 x_{i+1}\right) = 6i + 2$$

$$w_t\left(x_i y_i^1\right) = \begin{matrix} 2, \quad i = 1 \\ 6i + 1, \quad i > 1 \end{matrix}$$

$$w_t\left(y_i^1 z_i\right) = \begin{matrix} 3, \quad i = 1 \\ 6i, \quad i > 1 \end{matrix}$$

$$w_t\left(z_i u_i^1\right) = 6i - 2$$

$$w_t\left(u_i^1 z_{i+1}\right) = 6i - 1$$

Hence,

$$res \ \left(L_a^1\right) = \left\lceil \frac{6a - 4}{3} \right\rceil + 1.$$

Case 2. For $b = 2$

We define a vertex labeling

$$f(x_1)=0,\ f(z_1)=0,\ f\left(w_1^j\right)=2;\quad j=1,2, f\left(u_1^1\right)=2$$

$$f(x_{2i})=6i-2,\quad 1\le i\le \frac{a}{2},\quad a=2,4,6,\dots$$

$$f(x_{2i+1})=6i+2,\quad 1\le i\le \frac{a-2}{2},\quad a=4,6,8,\dots$$

$$f(z_{2i})=6i-2,\quad 1\le i\le \frac{a}{2},\quad a=2,4,6,\dots$$

$$f(z_{2i+1})=6i+2,\quad 1\le i\le \frac{a-2}{2},\quad a=4,6,8,\dots$$

$$f\left(w_{2i}^j\right)=6i,\quad 1\le i\le \frac{a}{2},\quad j=1,2,\quad a=2,4,6,\dots$$

$$f\left(w_{2i+1}^j\right)=6i+4,\quad 1\le i\le \frac{a-2}{2},\quad a=4,6,8,\dots$$

$$f\left(y_1^j\right)=0,\quad j=1,2$$

$$f\left(y_{2i}^j\right)=6i-2,\quad 1\le i\le \frac{a}{2},\quad j=1,2,\quad a=2,4,6,\dots$$

$$f\left(y_{2i+1}^j\right)=\ 6i+4,\quad 1\le i\le \frac{a-1}{2},\quad j=1,2,\quad a=3,5,7,\dots$$

$$f\left(u_{2i-1}^2\right)=6i-2,\quad 1\le i\le \frac{a-1}{2},\quad a=3,5,7,\dots$$

$$f\left(u_{2i}^2\right)=6i+2,\quad 1\le i\le \frac{a}{2},\quad a=2,4,6,\dots$$

$$f\left(u_{2i}^1\right)=6i,\quad 1\le i\le \frac{a}{2},\quad a=2,4,6,\dots$$

And we define an edge labeling

$$f\left(x_1w_1^1\right)=2,\ f\left(w_1^1w_1^2\right)=1,\ f\left(w_1^2x_2\right)=1, f\left(x_1y_1^1\right)=3$$

$$f\left(x_{2i}w_{2i}^1\right)=6i-3,\quad 1\le i\le \frac{a-1}{2},\quad a=3,5,7,\dots$$

$$f\left(w_{2i}^1w_{2i}^2\right)=6i-4,\quad 1\le i\le \frac{a-1}{2},\quad a=3,5,7,\dots$$

$$f\left(w_{2i+1}^1w_{2i+1}^2\right)=6i-3,\quad 1\le i\le \frac{a-2}{2},\quad a=4,6,8,\dots$$

$$f\left(w_{2i}^2x_{2i}\right)=6i-4,\quad 1\le i\le \frac{a-1}{2},\quad a=3,5,7,\dots$$

$$f\left(w_{2i+1}^2x_{2i+1}\right)=6i-1,\quad 1\le i\le \frac{a-2}{2},\quad a=4,6,8,\dots$$

$$f\left(x_{2i}y_{2i}^1\right)=6i-2,\quad 1\le i\le \frac{a}{2},\quad a=2,4,6,\dots$$

$$f\left(x_{2i+1}y_{2i+1}^1\right)=6i-1,\quad 1\le i\le \frac{a-1}{2},\quad a=3,5,7,\dots$$

$$f\left(y_1^1y_1^2\right)=2,\ f\left(y_1^2z_1\right)=1,\ f\left(z_1u_1^1\right)=4, f\left(u_1^1u_1^2\right)=2$$

$$f\left(y_{2i}^1y_{2i}^2\right)=6i-3,\quad 1\le i\le \frac{a}{2},\quad a=2,4,6,\dots$$

$$f\left(y_{2i+1}^1 y_{2i+1}^2\right) = 6i-2, \quad 1 \le i \le \frac{a-1}{2}, \quad a = 3,5,7,\dots$$

$$f\left(y_{2i}^2 z_{2i}\right) = 6i-4, \quad 1 \le i \le \frac{a}{2}, \quad a = 2,4,6,\dots$$

$$f\left(y_{2i+1}^2 z_{2i+1}\right) = 6i-3, \quad 1 \le i \le \frac{a-1}{2}, \quad a = 3,5,7,\dots$$

$$f\left(z_{2i} u_{2i}^1\right) = 6i-1, \quad 1 \le i \le \frac{a-1}{2}, \quad a = 3,5,7,\dots$$

$$f\left(z_{2i+1} u_{2i+1}^1\right) = 6i, \quad 1 \le i \le \frac{a-2}{2}, \quad a = 4,6,8,\dots$$

$$f\left(u_{2i}^1 u_{2i}^2\right) = 6i-3, \quad 1 \le i \le \frac{a-1}{2}, \quad a = 3,5,7,\dots$$

$$f\left(u_{2i+1}^1 u_{2i+1}^2\right) = 6i, \quad 1 \le i \le \frac{a-2}{2}, \quad a = 4,6,8,\dots$$

$$f\left(u_{2i-1}^2 z_{2i}\right) = 6i-5, \quad 1 \le i \le \frac{a}{2}, \quad a = 2,4,6,\dots$$

$$f\left(u_{2i}^2 z_{2i+1}\right) = 6i-4, \quad 1 \le i \le \frac{a-1}{2}, \quad a = 3,5,7,\dots$$

Edge weights are shown below which can be seen that all weights are distinct

$$w_t\left(x_i w_i^1\right) = 9i-5,\ w_t\left(w_i^1 w_i^2\right) = 9i-4, w_t\left(w_i^2 x_{i+1}\right) = 9i-2, w_t\left(x_i y_i^1\right) = 9i-6$$

$$w_t\left(y_i^1 y_i^2\right) = 9i-7, w_t\left(y_i^2 z_i\right) = 9i-8, w_t\left(z_i u_i^1\right) = 9i-3, w_t\left(u_i^1 u_i^2\right) = 9i+1\ , w_t\left(u_i^2 z_{i+1}\right) = 9i$$

Hence,

$$res\ \left(L_a^2\right) = \begin{cases} \left\lceil \frac{9a-6}{3} \right\rceil; & |E| \not\equiv 2,3\ (mod\ 6) \\ \left\lceil \frac{9a-6}{3} \right\rceil + 1; & |E| \equiv 2,3\ (mod\ 6) \end{cases}.$$

Case 3. For $b = 3$

We define a vertex labeling

$$f(x_1) = 0, f(z_1) = 0$$

$$f(x_{i+1}) = 4i+2, \quad 1 \le i \le a-1, \quad a = 2,3,4,\dots$$

$$f(z_{i+1}) = 4i+2, \quad 1 \le i \le a-1, \quad a = 2,3,4,\dots$$

$$f\left(w_i^1\right) = f\left(w_i^2\right) = 4i-2, \quad 1 \le i \le a-1, \quad a = 2,3,4,\dots$$

$$f\left(w_i^3\right) = 4i, \quad 1 \le i \le a-1, \quad a = 2,3,4,\dots$$

$$f\left(u_i^j\right) = 4i, \quad 1 \le i \le a-1, \quad j = 1,2,3, \quad a = 2,3,4,\dots$$

$$f\left(y_1^j\right) = 0, \quad j = 1,2,3$$

$$f\left(y_{i+1}^j\right) = 4i+2, \quad 1 \le i \le a-1, \quad j = 1,2,3, \quad a = 2,3,4,\dots$$

And we define an edge labeling

$$f\left(x_1 w_1^1\right) = 3,\ f\left(w_1^2 w_1^3\right) = 1,\ f\left(w_1^3 x_1\right) = 1,\ f\left(x_1 y_1^1\right) = 1, f\left(y_1^1 y_1^2\right) = 2, f\left(y_1^2 y_1^3\right) = 3,\ f\left(y_1^3 z_1\right) = 4,\ f\left(z_1 u_1^1\right) = 4$$

$$f\left(x_{i+1} w_{i+1}^1\right) = 4i+1, \quad 1 \le i \le a-2, \quad a = 3,4,5,\dots$$

$$f\left(w_i^1 w_i^2\right) = 4i-2, \quad 1 \le i \le a-1, \quad a = 2,3,4,\dots$$

$$f\left(w_{i+1}^2 w_{i+1}^3\right) = 4i+3, \quad 1 \le i \le a-2, \quad a = 3,4,5,\dots$$

$$f\left(w_{i+1}^{3}x_{i+2}\right)=4i-3,\quad 1\leq i\leq a-2,\quad a=3,4,5,\ldots$$
$$f\left(x_{i+1}y_{i+1}^{1}\right)=4i-3,\quad 1\leq i\leq a-1,\quad a=2,3,4,\ldots$$
$$f\left(y_{i+1}^{j}y_{i+1}^{j+1}\right)=4i+j-3,\quad j=1,2,\quad 1\leq i\leq a-1,\quad a=2,3,4,\ldots$$
$$f\left(y_{i+1}^{3}z_{i+1}\right)=4i,\quad 1\leq i\leq a-1,\quad a=2,3,4,\ldots$$
$$f\left(z_{i+1}u_{i+1}^{1}\right)=4i+2,\quad 1\leq i\leq a-2,\quad a=3,4,5,\ldots$$
$$f\left(u_{i}^{1}u_{i}^{2}\right)=4i,\quad 1\leq i\leq a-1,\quad a=2,3,4,\ldots$$
$$f\left(u_{i}^{3}z_{i+1}\right)=f\left(u_{i}^{2}u_{i}^{3}\right)=4i-2,\quad 1\leq i\leq a-1,\quad a=2,3,4,\ldots$$

Edge weights are shown below which can be seen that all weights are distinct

$$w_t\left(x_iw_i^1\right)=12i-7,\ w_t\left(w_i^1w_i^2\right)=12i-6,\ w_t\left(w_i^2w_i^3\right)=12i-5,\ w_t\left(w_i^3x_{i+1}\right)=12i-1$$
$$w_t\left(z_iu_i^1\right)=12i-4,\ w_t\left(u_i^1u_i^2\right)=12i-3,\ w_t\left(u_i^2u_i^3\right)=12i-2,\ w_t\left(u_i^3z_{i+1}\right)=12i$$
$$w_t\left(x_iy_i^1\right)=12i-11,\ w_t\left(y_i^1y_i^2\right)=12i-10,\ w_t\left(y_i^2y_i^3\right)=12i-9, w_t\left(y_i^3z_i\right)=12i-8$$

Hence,

$$res\ \left(L_a^3\right)=\ \left\lceil\frac{12a-8}{3}\right\rceil.$$

Case 4. For $b=4$
We define a vertex labeling

$$f\left(x_1\right)=0, f\left(z_1\right)=0,\ f\left(y_1^j\right)=0,\quad \forall j$$
$$f\left(x_{2i}\right)=10i-2,\quad 1\leq i\leq\frac{a-1}{2},\quad a=3,5,7,\ldots$$
$$f\left(z_{2i}\right)=10i-2,\quad 1\leq i\leq\frac{a-1}{2},\quad a=3,5,7,\ldots$$
$$f\left(x_{2i+1}\right)=10i+2,\quad 1\leq i\leq\frac{a-2}{2},\quad a=4,6,8,\ldots$$
$$f\left(z_{2i+1}\right)=10i+2,\quad 1\leq i\leq\frac{a-2}{2},\quad a=4,6,8,\ldots$$
$$f\left(w_{2i-1}^{j}\right)=10i-8,\quad 1\leq i\leq\frac{a}{2},\quad a=2,4,6,\ldots$$
$$f\left(w_{2i}^{j}\right)=10i-2,\quad 1\leq i\leq a-1,\quad \forall j,\quad a=2,3,4,\ldots$$
$$f\left(u_{2i-1}^{j}\right)=10i-6,\quad 1\leq i\leq\frac{a}{2},\quad a=2,4,6,\ldots$$
$$f\left(u_{2i}^{j}\right)=10i,\quad 1\leq i\leq\frac{a-1}{2},\quad a=3,5,7,\ldots$$
$$f\left(y_{2i}^{1}\right)=10i-4,\quad 1\leq i\leq\frac{a}{2},\quad a=2,4,6,\ldots$$
$$f\left(y_{2i+1}^{1}\right)=10i,\quad 1\leq i\leq\frac{a-1}{2},\quad a=3,5,7,\ldots$$
$$f\left(y_{2i}^{j}\right)=10i-2,\quad j>1,\quad 1\leq i\leq\frac{a}{2},\quad a=2,4,6,\ldots$$
$$f\left(y_{2i+1}^{j}\right)=10i+2,\quad j>1,\quad 1\leq i\leq\frac{a-1}{2},\quad a=3,5,7,\ldots$$

And we define an edge labeling

$$f\left(x_1w_1^1\right)=4, f\left(z_1u_1^1\right)=6, f\left(u_1^1u_1^2\right)=3, f\left(u_1^2u_1^3\right)=4, f\left(u_1^3u_1^4\right)=5, f\left(w_1^4x_2\right)=4,$$

$$f\left(x_3w_3^1\right)=12, f\left(w_3^4x_4\right)=10, f\left(x_1y_1^1\right)=1,$$

$$f\left(y_1^1y_1^2\right)=2, f\left(y_1^2y_1^3\right)=3, f\left(y_1^3y_1^4\right)=4, f\left(y_1^4z_1\right)=5$$

$$f\left(w_{2i-1}^jw_{2i-1}^{j+1}\right)=10i+j-8,\quad j=1,2,3,\quad 1\le i\le \frac{a}{2},\quad a=2,4,6,\ldots$$

$$f\left(x_{2i}w_{2i}^1\right)=10i-5,\quad 1\le i\le \frac{a-1}{2},\quad a=3,5,7,\ldots$$

$$f\left(w_{2i}^jw_{2i}^{j+1}\right)=10i+j-5,\quad j=1,2,3,\quad 1\le i\le \frac{a-1}{2},\quad a=3,5,7,\ldots$$

$$f\left(w_{2i}^4x_{2i+1}\right)=10i-5,\quad 1\le i\le \frac{a-1}{2},\quad a=3,5,7,\ldots$$

$$f\left(x_{2i+1}w_{2i+1}^1\right)=10i+2,\quad 1\le i\le \frac{a-2}{2},\quad a=4,6,8,\ldots$$

$$f\left(w_{2i+1}^4x_{2i+2}\right)=10i,\quad 1\le i\le \frac{a-2}{2},\quad a=4,6,8,\ldots$$

$$f\left(x_{2i}y_{2i}^1\right)=10i-8,\quad 1\le i\le \frac{a}{2},\quad a=2,4,6,\ldots$$

$$f\left(x_{2i+1}y_{2i+1}^1\right)=\quad 10i-1,\quad 1\le i\le \frac{a-1}{2},\quad a=3,5,7,\ldots$$

$$f\left(y_{2i+1}^jy_{2i+1}^{j+1}\right)=10i+6j-13,\quad j=1,2,\quad 1\le i\le \frac{a-1}{2},\quad a=3,5,7,\ldots$$

$$f\left(y_{2i}^jy_{2i}^{j+1}\right)=10i+j-10,\quad j=2,3,\quad 1\le i\le \frac{a}{2},\quad a=2,4,6,\ldots$$

$$f\left(y_{2i+1}^3y_{2i+1}^4\right)=10i,\quad 1\le i\le \frac{a-1}{2},\quad a=3,5,7,\ldots$$

$$f\left(y_{2i}^4z_{2i}\right)=10i-6,\quad 1\le i\le \frac{a}{2},\quad a=2,4,6,\ldots$$

$$f\left(y_{2i+1}^4z_{2i+1}\right)=10i+1,\quad 1\le i\le \frac{a-1}{2},\quad a=3,5,7,\ldots$$

$$f\left(u_{2i-1}^4z_{2i}\right)=10i-7,\quad 1\le i\le \frac{a}{2},\quad a=2,4,6,\ldots$$

$$f\left(z_{2i}u_{2i}^1\right)=10i-2,\quad 1\le i\le \frac{a-1}{2},\quad a=3,5,7,\ldots$$

$$f\left(u_{2i}^ju_{2i}^{j+1}\right)=\ 10i+j-4,\quad j=1,2,3,\quad 1\le i\le \frac{a-1}{2},\quad a=3,5,7,\ldots$$

$$f\left(u_{2i}^4z_{2i+1}\right)=10i-2,\quad 1\le i\le \frac{a-1}{2},\quad a=3,5,7,\ldots$$

$$f\left(z_{2i+1}u_{2i+1}^1\right)=10i+5,\quad 1\le i\le \frac{a-2}{2},\quad a=4,6,8,\ldots$$

$$f\left(u_{2i+1}^ju_{2i+1}^{j+1}\right)=10i+j+3,\quad j=1,2,3,\quad 1\le i\le \frac{a-2}{2},\quad a=4,6,8,\ldots$$

Edge weights are shown below which can be seen that all weights are distinct

$w_t\left(x_iw_i^1\right)=15i-9, w_t\left(w_i^1w_i^2\right)=15i-8, w_t\left(w_i^2w_i^3\right)=15i-7, w_t\left(w_i^3w_i^4\right)=15i-6, w_t\left(z_iu_i^1\right)=15i-5, w_t\left(u_i^1u_i^2\right)=15i-4, w_t\left(u_i^2u_i^3\right)=15i-3, w_t\left(u_i^3u_i^4\right)=15i-2, w_t\left(u_i^4z_{i+1}\right)=15i, w_t\left(x_iy_i^1\right)=15i-14, w_t\left(y_i^1y_i^2\right)=15i-13, w_t\left(y_i^2y_i^3\right)=15i-12, w_t\left(y_i^3y_i^4\right)=15i-11, w_t\left(y_i^4z_i\right)=15i-10,$

$w_t\left(w_i^4 x_{i+1}\right)=\begin{cases}15i+10; & i>1\\ 14; & i=1\end{cases}$ Hence,

$$res\ \left(L_a^4\right)=\begin{cases}\left\lceil\frac{15a-10}{3}\right\rceil; & |E|\not\equiv 2,3\ (mod\ 6)\\ \left\lceil\frac{15a-10}{3}\right\rceil+1; & |E|\equiv 2,3\ (mod\ 6)\end{cases}.$$

Case 5. For $b=5$
We define a vertex labeling

$$f\left(x_1\right)=0,\ f\left(z_1\right)=0,\ f\left(y_1^j\right)=0,\quad \forall j,\ f\left(u_i^j\right)=6i,\quad \forall i,j$$

$$f\left(x_{i+1}\right)=6i+2,\quad 1\leq i\leq a-2,\quad a=3,4,5,\ldots$$

$$f\left(z_{i+1}\right)=6i+2,\quad 1\leq i\leq a-2,\quad a=3,4,5,\ldots$$

$$f\left(w_1^j\right)=\begin{matrix}2, & j=1,2,5\\ 4, & j=3,4\end{matrix}$$

$$f\left(y_{i+1}^j\right)=6i+2,\quad 1\leq i\leq a-1,\quad \forall j,\quad a=2,3,4,\ldots$$

$$f\left(w_{i+1}^j\right)=\begin{matrix}6i+4, & 1\leq i\leq a-1, & a=2,3,4,\ldots, & j=1,2,5\\ 6i+6, & 1\leq i\leq a-1, & a=2,3,4,.., & j=3,4\end{matrix}$$

And we define an edge labeling

$$f\left(x_1 w_1^1\right)=5,\ f\left(x_1 y_1^1\right)=1$$

$$f\left(x_{i+1} w_{i+1}^1\right)=6i+1,\quad 1\leq i\leq a-2,\quad a=3,4,5,\ldots$$

$$f\left(w_1^1 w_1^2\right)=4, f\left(w_1^2 w_1^3\right)=3, f\left(w_1^3 w_1^4\right)=2, f\left(w_1^4 w_1^5\right)=5, f\left(w_1^5 x_2\right)=2$$

$$f\left(w_{i+1}^j w_{i+1}^{j+1}\right)=6i-j+1,\quad j=1,2,3,\quad 1\leq i\leq a-2,\quad a=3,4,5,\ldots$$

$$f\left(w_{i+1}^4 w_{i+1}^5\right)=6i+1,\quad 1\leq i\leq a-2,\quad a=3,4,5,\ldots$$

$$f\left(w_{i+1}^5 x_{i+2}\right)=6i,\quad 1\leq i\leq a-2,\quad a=3,4,5,\ldots$$

$$f\left(x_{i+1} y_{i+1}^1\right)=6i-3,\quad 1\leq i\leq a-1,\quad a=2,3,4,\ldots$$

$$f\left(y_1^1 y_1^2\right)=2, f\left(y_1^2 y_1^3\right)=3, f\left(y_1^3 y_1^4\right)=4, f\left(y_1^4 y_1^5\right)=5, f\left(y_1^5 z_1\right)=6$$

$$f\left(y_{i+1}^j y_{i+1}^{j+1}\right)=6i+j-3,\quad j=1,2,3,4,\quad 1\leq i\leq a-1,\quad a=2,3,4,\ldots$$

$$f\left(y_{i+1}^5 z_{i+1}\right)=6i+2,\quad 1\leq i\leq a-1,\quad a=2,3,4,\ldots$$

$$f\left(z_1 u_1^1\right)=7, f\left(u_1^1 u_1^2\right)=2, f\left(u_1^2 u_1^3\right)=3, f\left(u_1^3 u_1^4\right)=4, f\left(u_1^4 u_1^5\right)=5, f\left(u_1^5 x_2\right)=4$$

Edge weights are shown below which can be seen that all weights are distinct $w_t\left(x_i w_i^1\right)=18i-11, w_t\left(w_i^1 w_i^2\right)=18i-10, w_t\left(w_i^2 w_i^3\right)=18i-9, w_t\left(w_i^3 w_i^4\right)=18i-8, w_t\left(w_i^4 w_i^5\right)=18i-7, w_t\left(w_i^5 x_{i+1}\right)=18i-6, w_t\left(z_i u_i^1\right)=18i-5, w_t\left(u_i^1 u_i^2\right)=18i-4, w_t\left(u_i^2 u_i^3\right)=18i-3, w_t\left(u_i^3 u_i^4\right)=18i-2, w_t\left(u_i^4 u_i^5\right)=18i-1, w_t\left(u_i^5 z_{i+1}\right)=18i, w_t\left(x_i y_i^1\right)=18i-17, w_t\left(y_i^1 y_i^2\right)=18i-16, w_t\left(y_i^2 y_i^3\right)=18i-15, w_t\left(y_i^3 y_i^4\right)=18i-14, w_t\left(y_i^4 y_i^5\right)=18i-13, w_t\left(y_i^5 z_i\right)=18i-12$ Hence,

$$res\ \left(L_a^5\right)=\left\lceil\frac{18a-12}{3}\right\rceil.$$

□

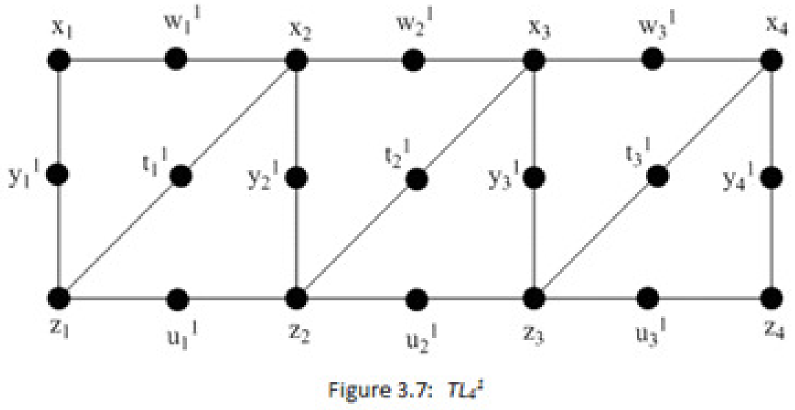

Figure 2: Subdivider triangular ladder graph

2.2. Reflexive edge strength of sub-divided triangular ladder graph

Let TL_a^1 be the one sub-division of the triangular ladder graph. It is formed by the vertex set $V\left(TL_a^1\right) = \{x_i,\ 1 \le i \le a\} \bigcup \{z_i,\ 1 \le i \le a\} \bigcup \{w_i^1,\ 1 \le i \le a-1\} \bigcup \{u_i^1,\ 1 \le i \le a-1\} \bigcup \{y_i^1,\ 1 \le i \le a\} \bigcup \{t_i^1,\ 1 \le i \le a-1\}$ and the edge set is $E\left(TL_a^1\right) = \{x_i w_i^1,\ 1 \le i \le a-1\} \bigcup \{w_i^1 x_{i+1},\ 1 \le i \le a-1\} \bigcup \{z_i u_i^1,\ 1 \le i \le a-1\} \bigcup \{u_i^1 z_{i+1},\ 1 \le i \le a-1\} \bigcup \{x_i y_i^1,\ 1 \le i \le a\} \bigcup \{y_i^1 z_i,\ 1 \le i \le a\} \bigcup \{x_{i+1} t_i^1,\ 1 \le i \le a-1\} \bigcup \{z_i t_i^1,\ 1 \le i \le a-1\}$. The cardinality of edges in TL_a^1 is $8a-6$. See Figure 2

Theorem 2.2. *Let TL_a^1 be the one sub-division of the triangular ladder graph, then*

$$res\ \left(TL_a^1\right) = \begin{cases} \left\lceil \frac{8a-6}{3} \right\rceil; & |E| \not\equiv 2,3\ (mod\ 6) \\ \left\lceil \frac{8a-6}{3} \right\rceil + 1; & |E| \equiv 2,3\ (mod\ 6) \end{cases}$$

Proof. Let TL_a^1 be the one sub division of the ladder graph then the lower bound of reflexive edge strength is given by Lemma 2.1 is

$$res\left(TL_a^1\right) \ge \begin{cases} \left\lceil \frac{|8a-6|}{3} \right\rceil; & |E| \not\equiv 2,3\ (mod\ 6) \\ \left\lceil \frac{8a-6}{3} \right\rceil + 1; & |E| \equiv 2,3\ (mod\ 6) \end{cases}$$

To show that

$$res\left(TL_a^1\right) \le \begin{cases} \left\lceil \frac{|8a-6|}{3} \right\rceil; & |E| \not\equiv 2,3\ (mod\ 6) \\ \left\lceil \frac{|8a-6|}{3} \right\rceil + 1; & |E| \equiv 2,3\ (mod\ 6) \end{cases}$$

We define a vertex labeling

$$f(x_1) = 0,\ f\left(u_1^1\right) = 0$$

$$f\left(x_{3i-1}\right) = 8i-4, \quad 1 \le i \le \frac{a+1}{3}, \quad a = 2,5,8,\ldots$$

$$f\left(x_{3i}\right) = 8i-2, \quad 1 \le i \le \frac{a}{3}, \quad a = 3,6,9,\ldots$$

$$f\left(x_{3i+1}\right) = 8i+2, \quad 1 \le i \le \frac{a-1}{3}, \quad a = 4,7,10,\ldots$$

$$f\left(z_{3i-1}\right) = 8i-4, \quad 1 \le i \le \frac{a+1}{3}, \quad a = 2,5,8,\ldots$$

$$f\left(z_{3i}\right) = 8i-2, \quad 1 \le i \le \frac{a}{3}, \quad a = 3,6,9,\ldots$$

$$f\left(z_{3i+1}\right)=8i+2,\quad 1\leq i\leq\frac{a-1}{3},\quad a=4,7,10,\ldots$$

$$f\left(y^1_{3i-1}\right)=8i-4,\quad 1\leq i\leq\frac{a+1}{3},\quad a=2,5,8,\ldots$$

$$f\left(y^1_{3i}\right)=8i-2,\quad 1\leq i\leq\frac{a}{3},\quad a=3,6,9,\ldots$$

$$f(y^1_{3i+1})=8i+2,\quad 1\leq i\leq\frac{a-1}{3},\quad a=4,7,10,\ldots$$

$$f\left(w^1_{3i-2}\right)=8i-6,\quad 1\leq i\leq\frac{a+1}{3},\quad a=2,5,8,\ldots$$

$$f\left(w^1_{3i-1}\right)=8i-4,\quad 1\leq i\leq\frac{a}{3},\quad a=3,6,9,\ldots$$

$$f(w^1_{3i})=8i,\quad 1\leq i\leq\frac{a-1}{3},\quad a=4,7,10,\ldots$$

$$f\left(u^1_{3i-1}\right)=8i-3,\quad 1\leq i\leq\frac{a}{3},\quad a=3,6,9,\ldots$$

$$f\left(u^1_{3i}\right)=8i,\quad 1\leq i\leq\frac{a-1}{3},\quad a=4,7,10,\ldots$$

$$f\left(u^1_{3i+1}\right)=8i+3,\quad 1\leq i\leq\frac{a-2}{3},\quad a=5,8,11,\ldots$$

$$f\left(t^1_{3i-2}\right)=8i-5,\quad 1\leq i\leq\frac{a+1}{3},\quad a=2,5,8,\ldots$$

$$f\left(t^1_{3i}-1\right)=8i-3,\quad 1\leq i\leq\frac{a}{3},\quad a=3,6,9,\ldots$$

$$f\left(t^1_{3i}\right)=8i+1,\quad 1\leq i\leq\frac{a-1}{3},\quad a=4,7,10,\ldots$$

And we define an edge labeling

$$f\left(x_1w^1_1\right)=3,\ f\left(z_1u^1_1\right)=3$$

$$f\left(x_{3i-1}w^1_{3i-1}\right)=8i-3,\quad 1\leq i\leq\frac{a}{3},\quad a=3,6,9,\ldots$$

$$f\left(x_{3i}w^1_{3i}\right)=8i-1,\quad 1\leq i\leq\frac{a-1}{3},\quad a=4,7,10,\ldots$$

$$f\left(x_{3i+1}w^1_{3i+1}\right)=8i+1,\quad 1\leq i\leq\frac{a-2}{3},\quad a=5,8,11,\ldots$$

$$f\left(z_{3i-1}u^1_{3i-1}\right)=8i-6,\quad 1\leq i\leq\frac{a}{3},\quad a=3,6,9,\ldots$$

$$f\left(z_{3i}u^1_{3i}\right)=8i-3,\quad 1\leq i\leq\frac{a-1}{3},\quad a=4,7,10,\ldots$$

$$f\left(z_{3i+1}u^1_{3i+1}\right)=8i-2,\quad 1\leq i\leq\frac{a-2}{3},\quad a=5,8,11,\ldots$$

$$f\left(w^1_{3i-2}x_{3i-1}\right)=8i-7,\quad 1\leq i\leq\frac{a+1}{3},\quad a=2,5,8,\ldots$$

$$f\left(w^1_{3i-1}x_{3i}\right)=8i-3,\quad 1\leq i\leq\frac{a}{3},\quad a=3,6,9,\ldots$$

$$f\left(w^1_{3i}x_{3i+1}\right)=8i-3,\quad 1\leq i\leq\frac{a-1}{3},\quad a=4,7,10,\ldots$$

$$f\left(u^1_1x_2\right)=2,\quad f\left(x_1y^1_1\right)=1,\ f\left(y^1_1z_1\right)=2$$

$$f\left(u^1_{3i-1}z_{3i}\right)=8i-5,\quad 1\le i\le \frac{a}{3},\quad a=3,6,9,\ldots$$

$$f\left(u^1_{3i}z_{3i+1}\right)=8i-4,\quad 1\le i\le \frac{a-1}{3},\quad a=4,7,10,\ldots$$

$$f\left(u^1_{3i+1}z_{3i+2}\right)=8i-1,\quad 1\le i\le \frac{a-2}{3},\quad a=5,8,11,\ldots$$

$$f\left(x_{3i-1}y^1_{3i-1}\right)=8i-7,\quad 1\le i\le \frac{a+1}{3},\quad a=2,5,8,\ldots$$

$$f\left(x_{3i}y^1_{3i}\right)=8i-3,\quad 1\le i\le \frac{a}{3},\quad a=3,6,9,\ldots$$

$$f\left(x_{3i+1}y^1_{3i+1}\right)=8i-3,\quad 1\le i\le \frac{a-1}{3},\quad a=4,7,10,\ldots$$

$$f\left(y^1_{3i-1}z_{3i-1}\right)=8i-6,\quad 1\le i\le \frac{a+1}{3},\quad a=2,5,8,\ldots$$

$$f\left(y^1_{3i}z_{3i}\right)=8i-2,\quad 1\le i\le \frac{a}{3},\quad a=3,6,9,\ldots$$

$$f\left(y^1_{3i+1}z_{3i+1}\right)=8i-2,\quad 1\le i\le \frac{a-1}{3},\quad a=4,7,10,\ldots$$

$$f\left(x_{3i-1}t^1_{3i-2}\right)=8i-7,\quad 1\le i\le \frac{a+1}{3},\quad a=2,5,8,\ldots$$

$$f\left(x_{3i}t^1_{3i-1}\right)=8i-3,\quad 1\le i\le \frac{a}{3},\quad a=3,6,9,\ldots$$

$$f\left(x_{3i+1}t^1_{3i}\right)=8i-3,\quad 1\le i\le \frac{a-1}{3},\quad a=4,7,10,\ldots$$

$$f\left(z_{3i-2}t^1_{3i-2}\right)=8i-7,\quad 1\le i\le \frac{a+1}{3},\quad a=2,5,8,\ldots$$

$$f\left(z_{3i-1}t^1_{3i-1}\right)=8i-5,\quad 1\le i\le \frac{a+1}{3},\quad a=3,6,9,\ldots$$

$$f\left(z_{3i}t^1_{3i}\right)=8i-3,\quad 1\le i\le \frac{a-1}{3},\quad a=4,7,11,\ldots$$

Edge weights are shown below which can be seen that all weights are distinct

$$w_t\left(x_iw^1_i\right)=8i-3, w_t\left(w^1_ix_{i+1}\right)=8i-1, w_t\left(z_iu^1_i\right)=8i-5$$

$$w_t\left(u^1_iz_{i+1}\right)=8i-2, w_t\left(x_iy^1_i\right)=8i-7, w_t\left(y^1_iz_i\right)=8i-6, w_t\left(x_{i+1}t^1_i\right)=8i, w_t\left(z_it^1_i\right)=8i-4$$

Hence,

$$res\ \left(TL^1_a\right)=\begin{cases}\left\lceil\frac{8a-6}{3}\right\rceil; & |E|\not\equiv 2,3\ (mod\ 6)\\ \left\lceil\frac{8a-6}{3}\right\rceil+1; & |E|\equiv 2,3\ (mod\ 6)\end{cases}.$$

□

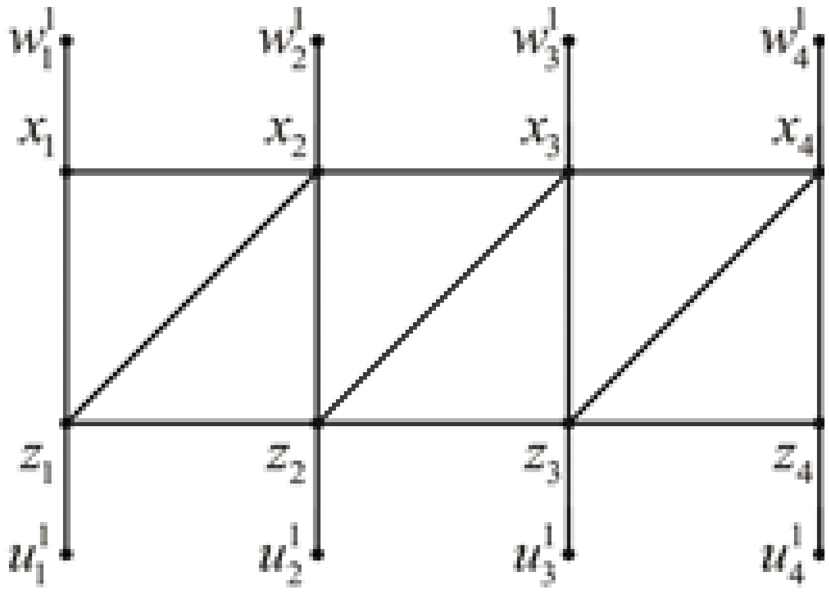

Figure 3: PTL_4

2.3. Reflexive edge strength with pendent edges of triangular ladder graph

Let PTL_a be the pendent edges of the triangular ladder graph. It is formed by the vertex set $V\left(PTL_a\right) = x_i,\ \ 1 \leq i \leq a-1,\ \ a \geq 2 \bigcup z_i,\ \ 1 \leq i \leq a-1,\ \ a \geq 2 \bigcup w_i^1,\ \ 1 \leq i \leq a,\ \ a \geq 1 \bigcup z_i,\ \ 1 \leq i \leq a,\ \ a \geq 1$ and the edge set is $E\left(PTL_a\right) = \{x_i x_{i+1},\ \ 1 \leq i \leq a-1,\ \ a \geq 2\} \bigcup \{z_i z_{i+1},\ \ 1 \leq i \leq a-1,\ \ a \geq 2\} \bigcup \{x_{i+1} z_i,\ \ 1 \leq i \leq a-1,\ \ a \geq 2\} \bigcup \{x_i w_i^1,\ \ 1 \leq i \leq a,\ \ a \geq 1\} \bigcup \{z_i u_i^1,\ \ 1 \leq i \leq a,\ \ a = 1, 2, 3, \ldots\}$. Cardinality of edges in PTL_a is $6a-3$. See Figure 3.

Theorem 2.3. *Let PTL_a be the pendent edges of the triangular ladder graph, then*

$$res\ \ (PTL_a) = \begin{cases} \left\lceil \frac{6a-3}{3} \right\rceil; & |E| \not\equiv 2, 3\ (mod\ 6) \\ \left\lceil \frac{6a-3}{3} \right\rceil + 1; & |E| \equiv 2, 3\ (mod\ 6) \end{cases}$$

Proof. Let PTL_a be the pendent edges of triangular ladder graph then the lower bound of reflexive edge strength is given by Lemma 2.1 is

$$res\left(PTL_a\right) \geq \begin{cases} \left\lceil \frac{|6a-3|}{3} \right\rceil; & |E| \not\equiv 2, 3\ (mod\ 6) \\ \left\lceil \frac{6a-3}{3} \right\rceil + 1; & |E| \equiv 2, 3\ (mod\ 6) \end{cases}$$

To show that

$$res\left(TL_a^1\right) \leq \begin{cases} \left\lceil \frac{|6a-3|}{3} \right\rceil; & |E| \not\equiv 2, 3\ (mod\ 6) \\ \left\lceil \frac{|6a-3|}{3} \right\rceil + 1; & |E| \equiv 2, 3\ (mod\ 6) \end{cases}$$

We define a vertex labeling

$$f\left(x_1\right) = 1,\ f\left(z_1\right) = 1,\ \ f\left(w_1^1\right) = 0$$

$$f\left(z_{i+1}\right) = f\left(x_{i+1}\right) = 2i-1,\ \ 1 \leq i \leq a-1,\ \ a = 2, 3, 4, \ldots$$

$$f\left(w_{i+1}^1\right) = 2i+2,\ \ 1 \leq i \leq a-1,\ \ a = 2, 3, 4, \ldots$$

$$f\left(u_i^1\right) = 2i,\ \ 1 \leq i \leq a,\ \ a = 1, 2, 3, \ldots$$

And we define an edge labeling

$$f\left(x_1 x_2\right) = 2,\ f\left(x_1 z_1\right) = 2,\ f\left(x_1 w_1^1\right) = 1,\ f\left(z_1 u_1^1\right) = 1$$

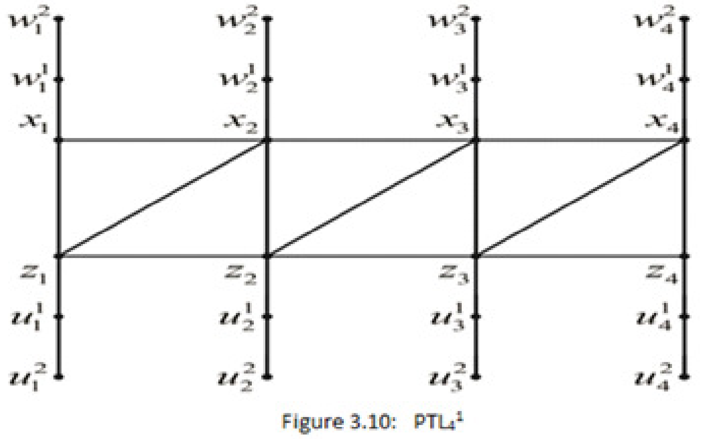

Figure 4: PTL_4^1

$$f\left(x_{i+1}x_{i+2}\right)=2i,\quad 1\le i\le a-2,\quad a=3,4,5,\ldots$$
$$f\left(z_iz_{i+1}\right)=2i,\quad 1\le i\le a-1,\quad a=2,3,4,\ldots$$
$$f\left(x_{i+1}z_{i+1}\right)=2i,\quad 1\le i\le a-1,\quad a=2,3,4,\ldots$$
$$f\left(x_{i+1}w_{i+1}^1\right)=f\left(z_{i+1}u_{i+1}^1\right)=f\left(x_{i+1}z_i\right)=2i-1,\quad 1\le i\le a-1,\quad a=2,3,4,\ldots$$

Edge weights are shown below which can be seen that all weights are distinct

$$w_t\left(x_ix_{i+1}\right)=6i, w_t\left(z_iz_{i+1}\right)=6i-2, w_t\left(x_iz_i\right)=6i-4, w_t\left(x_iw_i^1\right)=6i-5, w_t\left(z_iu_i^1\right)=6i-3, w_t\left(x_{i+1}z_i\right)=6i-1$$

Hence,

$$res\ \left(PTL_a\right)=\begin{cases}\left\lceil\frac{6a-3}{3}\right\rceil; & |E|\not\equiv 2,3\ (mod\ 6)\\ \left\lceil\frac{6a-3}{3}\right\rceil+1; & |E|\equiv 2,3\ (mod\ 6)\end{cases}.$$

□

2.4. Reflexive edge strength of sub-division of triangular ladder graph with pendent edges

Let PTL_a^1 be the one sub division of pendent edges of the triangular ladder graph. It is formed by the vertex set

$V\left(PTL_a^1\right)=\{x_i,\ 1\le i\le a-1,\ a=2,3,4,\ldots\}\bigcup\{z_i,\ 1\le i\le a-1,\ a=2,3,4,\ldots\}\bigcup\{w_i^j,\ j=1,2,\ 1\le i\le a,\ a=1,2,3,\ldots\}\bigcup\{u_i^j,\ j=1,2,\ 1\le i\le a,\ a=1,2,3,\ldots\}$ and the edge set $E\left(PTL_a^1\right)=\{x_ix_{i+1},\ 1\le i\le a-1,\ a=2,3,4,\ldots\}\bigcup\{z_iz_{i+1},\ 1\le i\le a-1,\ a=2,3,4,\ldots\}\bigcup\{x_{i+1}z_i,\ 1\le i\le a-1,\ a=2,3,4,\ldots\}\bigcup\{x_iw_i^1,\ 1\le i\le a,\ a=1,2,3,\ldots\}\bigcup\{w_i^1w_i^2,\ 1\le i\le a,\ a=1,2,3,\ldots\}\bigcup\{z_iu_i^1,\ 1\le i\le a,\ a=1,2,3,\ldots\}\bigcup\{u_i^1u_i^2,\ 1\le i\le a,\ a=1,2,3,\ldots\}$. Cardinality of edges in PTL_a^1 is $8a-3$, see Figure 4.

Theorem 2.4. *Let PTL_a^1 be the one sub division of pendent edges of the triangular ladder graph, then*

$$res\ \left(PTL_a^1\right)=\begin{cases}\left\lceil\frac{8a-3}{3}\right\rceil; & |E|\not\equiv 2,3\ (mod\ 6)\\ \left\lceil\frac{8a-3}{3}\right\rceil+1; & |E|\equiv 2,3\ (mod\ 6)\end{cases}$$

Proof. Let PTL_a^1 be the one sub division of the pendent edges of triangular ladder graph then the lower bound of reflexive edge strength is given by Lemma 2.1 is

$$res\left(PTL_a^1\right)\ge\begin{cases}\left\lceil\frac{|8a-3|}{3}\right\rceil; & |E|\not\equiv 2,3\ (mod\ 6)\\ \left\lceil\frac{8a-3}{3}\right\rceil+1; & |E|\equiv 2,3\ (mod\ 6)\end{cases}$$

To show that

$$res\left(PTL_a^1\right) \leq \begin{cases} \left\lceil \frac{|8a-3|}{3} \right\rceil; & |E| \not\equiv 2,3\ (mod\ 6) \\ \left\lceil \frac{|8a-3|}{3} \right\rceil + 1; & |E| \equiv 2,3\ (mod\ 6) \end{cases}$$

We define a vertex labeling

$$f(x_1) = 0,\ f(z_1) = 0$$

$$f(x_{2i}) = 5i,\quad 1 \leq i \leq \frac{a}{2},\quad a = 2,4,6,\dots$$

$$f(x_{2i+1}) = 5i+3,\quad 1 \leq i \leq \frac{a-1}{2},\quad a = 3,5,7,\dots$$

$$f(z_{2i}) = 5i,\quad 1 \leq i \leq \frac{a}{2},\quad a = 2,4,6,\dots$$

$$f\left(w_{2i+1}^1\right) = 5i+2,\quad 1 \leq i \leq \frac{a-1}{2},\quad a = 3,5,7,\dots$$

$$f\left(w_{2i}^2\right) = 5i-2,\quad f\left(w_{2i}^1\right) = 5i-1,\quad 1 \leq i \leq \frac{a}{2},\quad a = 2,4,6,\dots$$

$$f\left(w_{2i+1}^2\right) = 5i+1,\ f(z_{2i+1}) = 5i+3,\quad 1 \leq i \leq \frac{n-1}{2},\quad n = 3,5,7,\dots$$

$$f\left(u_1^1\right) = 0,\quad f\left(u_1^2\right) = 0, f\left(w_1^2\right) = 0, f\left(w_1^1\right) = 0$$

$$f\left(u_{2i}^j\right) = 5i,\quad 1 \leq i \leq \frac{a}{2},\quad j = 1,2,\quad a = 2,4,6,\dots$$

$$f\left(u_{2i+1}^j\right) = 5i+3,\quad j = 1,2,\quad 1 \leq i \leq \frac{a-1}{2},\quad a = 3,5,7,\dots$$

And we define an edge labeling

$$f(x_{i+2}x_{i+3}) = 5i-1,\quad 1 \leq i \leq a-3,\quad a = 4,5,6,\dots$$

$$f(x_{i+3}x_{i+4}) = 5i+2,\quad 1 \leq i \leq a-4,\quad a = 5,6,7,\dots$$

$$f(z_1z_2) = 3, f(z_2z_3) = 3, f(x_1x_2) = 1, f(x_2x_3) = 1$$

$$f(z_{i+2}z_{i+3}) = 5i+1,\quad 1 \leq i \leq a-3,\quad a = 4,5,6,\dots$$

$$f(z_{i+3}z_{i+4}) = 5i+4,\quad 1 \leq i \leq a-4,\quad a = 5,6,7,\dots$$

$$f\left(x_1w_1^1\right) = 2, f\left(x_2w_2^1\right) = 1, f\left(z_1u_1^1\right) = 3, f\left(z_2u_2^1\right) = 3, f(x_1z_1) = 3$$

$f\left(x_{i+2}w_{i+2}^1\right) = -2+5i,\quad f\left(x_{i+3}w_{i+3}^1\right) = 2+5i,\quad f\left(z_{i+2}u_{i+2}^1\right) = 5i+1,\quad f\left(z_{i+3}u_{i+3}^1\right) = 5i+4,\quad 1 \leq i \leq a-3,\quad a = 4,5,6,\dots$

$$f(x_{3i+1}z_{3i+1}) = 8i-1,\quad 1 \leq i \leq \frac{a+1}{3},\quad a = 4,7,10,\dots$$

$$f(x_{3i-1}z_{3i-1}) = 8i-7,\quad 1 \leq i \leq \frac{a+1}{3},\quad a = 2,5,8,\dots$$

$$f(x_{3i}z_{3i}) = 8i-5,\quad 1 \leq i \leq \frac{a}{3},\quad a = 3,6,9,\dots$$

$$f\left(w_1^1w_1^2\right) = 1, f\left(u_1^1u_1^2\right) = 5$$

$$f\left(w_{3i-1}^1w_{3i-1}^2\right) = 8i-6,\quad 1 \leq i \leq \frac{a+1}{3},\quad a = 2,5,8,\dots$$

$$f\left(w_{3i}^1w_{3i}^2\right) = 8i-4,\quad 1 \leq i \leq \frac{a}{3},\quad a = 3,6,9,\dots$$

$$f\left(w_{3i+1}^{1}w_{3i+1}^{2}\right) = 8i, \quad 1 \le i \le \frac{a-1}{3}, \quad a = 4,7,10,\dots$$
$$f\left(u_{3i-1}^{1}u_{3i-1}^{2}\right) = -5+8i, \quad 1 \le i \le \frac{a+1}{3}, \quad a = 2,5,8,\dots$$
$$f\left(u_{3i}^{1}u_{3i}^{2}\right) = -3+8i, \quad 1 \le i \le \frac{a}{3}, \quad a = 3,6,9,\dots$$
$$f\left(u_{3i+1}^{1}u_{3i+1}^{2}\right) = 1+8i, \quad 1 \le i \le \frac{a-1}{3}, \quad a = 4,7,10,\dots$$

Edge weights are shown below which can be seen that all weights are distinct

$$w_t\left(x_i x_{i+1}\right) = 8i-2, w_t\left(z_i z_{i+1}\right) = 8i, w_t\left(x_i z_i\right) = 8i-5, \; w_t\left(w_i^1 w_i^2\right) = 8i-7$$
$$w_t\left(x_i w_i^1\right) = 8i-6, w_t\left(u_i^1 u_i^2\right) = 8i-3, w_t\left(z_i u_i^1\right) = 6i-4, w_t\left(x_{i+1} z_i\right) = 8i-1$$

Hence,

$$res\ \left(PTL_a^1\right) = \begin{cases} \left\lceil \frac{8a-3}{3} \right\rceil; & |E| \not\equiv 2,3 \ (mod\ 6) \\ \left\lceil \frac{8a-3}{3} \right\rceil + 1; & |E| \equiv 2,3 \ (mod\ 6) \end{cases}.$$

□

3. Conclusion

In this paper, we determine the previse value of reflexive edge strength of different families of subdivided ladder graphs. In particular subdivided ladder graph L_a^b, subdivided triangular ladder graph TL_a^1, and triangular ladder graph PTL_a^1 with pendant vertices.

References

[1] Chartrand, G., Jacobson, M. S., Lehel, J., Oellermann, O. R., Ruiz, S., Saba, F. (1988). Irregular networks. *Congr. Numer,* 64(197-210), 250th. 1

[2] Al-Mushayt, O. B. S., Ahmad, A., Siddiqui, M. K. (2012). On the total edge irregularity strength of hexagonal grid graphs. *Australas. J Comb.,* 53, 263-272. 1

[3] Amar, D., Togni, O. (1998). Irregularity strength of trees. *Discrete Math.*, 190(1-3), 15-38. 1

[4] Bača, M., Miller, M., Ryan, J. (2007). On irregular total labellings. *Discrete Math.*, 307(11-12), 1378-1388. 1

[5] Bača, M., Siddiqui, M. K. (2014). Total edge irregularity strength of generalized prism. *Appl. Math. Comput.*, 235, 168-173. 1

[6] Ivančo, J., Jendrol', S. (2006). Total edge irregularity strength of trees. *Dicuss Math Graph T.,* 26(3), 449-456. 1

[7] Tanna, D., Ryan, J., Semaničová-Feňovčíková, A. (2017). Edge irregular reflexive labeling of prisms and wheels. *Australas. J. Comb*, 69(3), 394-401. 1

[8] Agustin, I. H., Utoyo, I., Venkatachalam, M. (2020, May). Edge irregular reflexive labeling of some tree graphs. In Journal of Physics: Conference Series (Vol. 1543, No. 1, p. 012008). IOP Publishing. 1

[9] Indriati, D., Rosyida, I. (2020, March). Edge irregular reflexive labeling on corona of path and other graphs. In Journal of Physics: Conference Series (Vol. 1489, No. 1, p. 012004). IOP Publishing. 1

[10] Indriati, D., Rosyida, I. (2020, March). Edge irregular reflexive labeling on corona of path and other graphs. In Journal of Physics: Conference Series (Vol. 1489, No. 1, p. 012004). IOP Publishing. 1

[11] Yoong, K. K., Hasni, R., Irfan, M., Taraweh, I., Ahmad, A., Lee, S. M. (2021). On the edge irregular reflexive labeling of corona product of graphs with path. *AKCE Int. J. Graphs Comb.,* 18(1), 53-59. 1

[12] Budi, N. I. S., Indriati, D., Winarno, B. (2021, February). Edge irregular reflexive labeling on tadpole graphs Tm, 1 and Tm, 2. In AIP Conference Proceedings (Vol. 2326, No. 1). AIP Publishing. 1

[13] Junetty, R., Indriati, D., Winarno, B. (2022, November). Edge irregular reflexive labeling of palm tree graph C3-B2, r and C3- B3, r. In AIP Conference Proceedings (Vol. 2566, No. 1). AIP Publishing. 1

[14] Novelia, J. A., Indriati, D. (2021, February). Edge irregular reflexive labeling on banana tree graphs B2, n and B3, n. In AIP Conference Proceedings (Vol. 2326, No. 1). AIP Publishing. 1

[15] Agustin, I. H., Utoyo, M. I., Venkatachalam, M. (2021). The reflexive edge strength on some almost regular graphs. *Heliyon,* 7(5). 1

[16] Setiawan, I., Indriati, D. (2021). Edge irregular reflexive labeling on sun graph and corona of cycle and null graph with two vertices. *Indonesian Journal of Combinatorics*, 5(1), 35-45. 1