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Hamiltonicity in directed Toeplitz graphs having increasing edges of length 1, 3 and 7

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Abstract

A directed Toeplitz graph $T_n\langle a_1, \ldots, a_p; b_1, \ldots, b_q\rangle$ with vertices $1, 2, \ldots, n$, where the edge (i, j) occurs if and only if $j - i = a_s$ or $i - j = b_t$ for some $1 \leq s \leq p$ and $1 \leq t \leq q$, is a digraph whose adjacency matrix is a Toeplitz matrix. In this paper, we study hamiltonicity in directed Toeplitz graphs having increasing edges of length 1, 3 and 7, only.

Keywords: Adjacency matrix; Toeplitz graph; Hamiltonian graph.
2010 MSC: Primary 05C20; Secondary 05C45.

1. Introduction

A directed or undirected Toeplitz graph is a graph whose adjacency matrix is a Toeplitz matrix, a square matrix which has constant values along all diagonals parallel to the main diagonal. Toeplitz matrices arise in a many problems in engineering and applied mathematics, for example in signal processing, queuing theory, time series analysis, integral equations, etc.

A directed Toeplitz graph $T_n\langle a_1, \ldots, a_p; b_1, \ldots, b_q\rangle$ is a diagraph of order $n > max\{a_p, b_q\}$, with vertices $1, 2, \ldots, n$, where the edge (i, j) occurs if and only if the increasing edges (the edges of the type (i, j) where $i < j$) and decreasing edges (the edges of the type (i, j) where $i > j$) are of length a_s and b_t, respectively, for some $1 \leq s \leq p$ and $1 \leq t \leq q$. We consider finite simple directed graphs.

Properties of Toeplitz graphs, such as colourability, planarity, bipartiteness, connectivity, cycle discrepancy, edge irregularity strength, decomposition, labeling, and metric dimension have been studied in [1]-[6], [8]-[12], [14]-[15], and [27]. Hamiltonian properties of Toeplitz graphs were first investigated by R. van Dal et al. in [7] and then studied in [13, 26, 28], while the hamiltonicity in directed Toeplitz graphs was first studied by S. Malik and T. Zamfirescu in [25], by S. Malik in [16], by S. Malik and A.M. Qureshi in [24], by S. Malik in [17]-[22], and then by S. Malik and A.M. Qureshi in [23].

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For $a_1 = 1$ and $a_2 = 3$, in [20] and [21], the hamiltonicity of Toeplitz graphs with $a_3 = 4$ was investigated, while in [23], it was investigated for Toeplitz graphs with $a_3 = 5$, and for $a_3 = 6$ it is under review in a paper. In this paper we still keep $a_1 = 1$ and $a_2 = 3$ but then we consider $a_3 = 7$, that is, we investigate the hamiltonicity in Toeplitz graphs $T_n\langle 1, 3, 7; b\rangle$.

For a vertex v of $T_n\langle 1, 3, 7; b\rangle$, we define paths $A_{v\to v-10}$, $B_{v\to v+10}$, $C_{v\to v+4}$, and $D_{v\to v-7}$ in $T_n\langle 1, 3, 7; b\rangle$ as $A_{v\to v-10} = (v, v-3, v-6, v-9, v-2, v-5, v-4, v-7, v-10)$, $B_{v\to v+10} = (v, v+3, v+10)$, $C_{v\to v+4} = (v, v+3, v+4)$, and $D_{v\to v-7} = (v, v-2, v-4, v-6, v-3, v-5, v-7)$, respectively, see Figure 1.

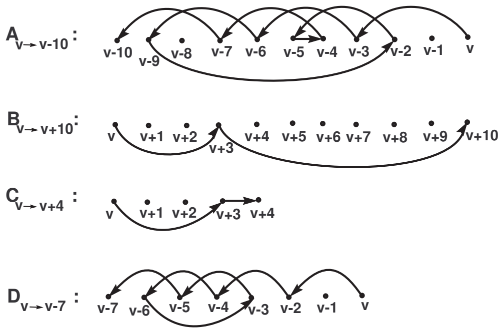

Figure 1: Paths $A_{v\to v-10}$, $B_{v\to v+10}$, $C_{v\to v+4}$ and $D_{v\to v-7}$ in $T_n\langle 1, 3, 7; b\rangle$

We underline a pair of consecutive vertices (say $n-2$ and $n-1$) as $(\underline{n-2, n-1})$ to emphasize that $(n-2, n-1)$ is an edge in the hamiltonian cycle. Note that $C_{n-5\to n-1} = (n-5, \underline{n-2, n-1})$.

Remark 1: If the Toeplitz graph $T_n\langle 1, 3, 7; b\rangle$ has a hamiltonian cycle containing the edge $(n-2,\ n-1)$, then $T_{n+(b-1)}\langle 1, 3, 7; b\rangle$ enjoys the same property. Because such a hamiltonian cycle in $T_n\langle 1, 3, 7; b\rangle$ can be transformed into a hamiltonian cycle in $T_{n+(b-1)}\langle 1, 3, 7; b\rangle$, by replacing the edge $(n-2,\ n-1)$ with the path $(n-2, n+1, n+2, \ldots, \underline{n+(b-3), n+(b-2)}, n+(b-1), n-1)$, which preserves the same property. For example, see Figure 2, where a hamiltonian cycle in $T_8\langle 1, 3, 7; 6\rangle$ is transformed into a hamiltonian cycle in $T_{13}\langle 1, 3, 7; 6\rangle$ by replacing the edge $(6, 7)$ with the path $(6, 9, 10, \underline{11, 12}, 13, 7)$, which preserves the same property so $T_{14}\langle 1, 3, 7; 5\rangle$ can be transformed into a hamiltonian cycle in $T_{18}\langle 1, 3, 7; 6\rangle$, and so on.

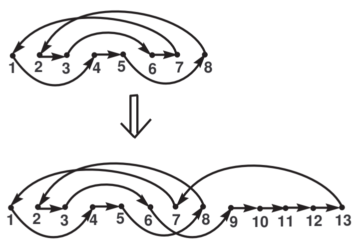

Figure 2: Hamiltonian cycles in $T_8\langle 1, 3, 7; 6\rangle$ and $T_{13}\langle 1, 3, 7; 6\rangle$

2. Toeplitz Graphs $T_n\langle 1, 3, 7; b\rangle$ for odd b

In this section, we will discuss the hamiltonicity in Toeplitz Graphs $T_n\langle 1, 3, 7; b\rangle$ for odd b. For $b = 1$, clearly $T_n\langle 1, 3, 7; 1\rangle$ is hamiltonian if and only if $n = 8$, because the decreasing edges are of length one only and this is only possible when $n = 8$ and it is easily seen that $T_8\langle 1, 3, 7; 1\rangle$ has the unique hamiltonian cycle $(1, 8, 7, 6, 5, 4, 3, 2, 1)$.

Now we will investigate the hamiltonicity in $T_n\langle 1, 3, 7; b\rangle$ for odd $b > 1$.

Theorem 2.1. *$T_n\langle 1,3,7;3\rangle$ is hamiltonian if and only if n is even.*

Proof. Let n be even. Clearly $n \geq 8$.

If $n \cong 0\, mod\, 10$, then a hamiltonian cycle in $T_n\langle 1,3,7;3\rangle$ is $(1,2,9) \cup B_{9\to 19} \cup B_{19\to 29} \cup \cdots \cup B_{n-11\to n-1} \cup (n-1,n) \cup A_{n\to n-10} \cup A_{n-10\to n-20} \cup \cdots \cup A_{20\to 10} \cup (10,7,8,5,6,3,4,1)$, see Figure 3.

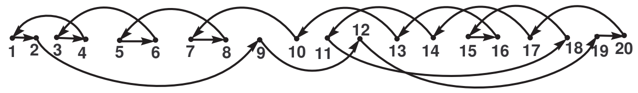

Figure 3: Hamiltonian cycle in $T_{20}\langle 1,3,7;3\rangle$

If $n \cong 2\, mod\, 10$, then a hamiltonian cycle in $T_n\langle 1,3,7;3\rangle$ is $(1,2,5,8,11) \cup B_{11\to 21} \cup B_{21\to 31} \cup \cdots \cup B_{n-11\to n-1} \cup (n-1,n) \cup A_{n\to n-10} \cup A_{n-10\to n-20} \cup \cdots \cup A_{22\to 12} \cup (12,9,6,3,10,7,4,1)$, see Figure 4.

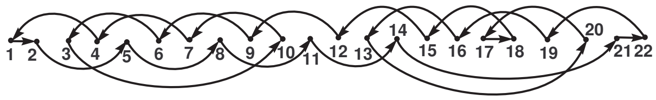

Figure 4: Hamiltonian cycle in $T_{22}\langle 1,3,7;3\rangle$

If $n \cong 4\, mod\, 10$, then a hamiltonian cycle in $T_n\langle 1,3,7;3\rangle$ is $(1,2,3) \cup B_{3\to 13} \cup B_{13\to 23} \cup \cdots \cup B_{n-11\to n-1} \cup (n-1,n) \cup A_{n\to n-10} \cup A_{n-10\to n-20} \cup \cdots \cup A_{14\to 4} \cup (4,1)$, see Figure 5.

If $n \cong 6\, mod\, 10$, then a hamiltonian cycle in $T_n\langle 1,3,7;3\rangle$ is $(1,2,5) \cup B_{5\to 15} \cup B_{15\to 25} \cup \cdots \cup B_{n-11\to n-1} \cup (n-1,n) \cup A_{n\to n-10} \cup A_{n-10\to n-20} \cup \cdots \cup A_{16\to 6} \cup (6,3,4,1)$, see Figure 6.

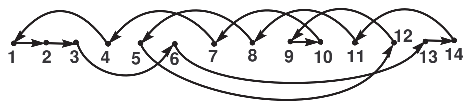

Figure 5: Hamiltonian cycle in $T_{14}\langle 1,3,7;3\rangle$

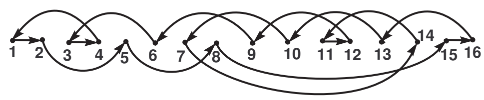

Figure 6: Hamiltonian cycle in $T_{16}\langle 1,3,7;3\rangle$

If $n \cong 8\, mod\, 10$ and $n \neq 8$, then a hamiltonian cycle in $T_n\langle 1,3,7;3\rangle$ is $(1,2,3,10,17) \cup B_{17\to 27} \cup B_{27\to 37} \cup \cdots \cup B_{n-11\to n-1} \cup (n-1,n) \cup A_{n\to n-10} \cup A_{n-10\to n-20} \cup \cdots \cup A_{18\to 8} \cup (8,5,6,7,4,1)$, see Figure 7. And a hamiltonian cycle in $T_8\langle 1,3,7;3\rangle$ is $(1,8,5,2,3,6,7,4)$.

Conversely, $T_n\langle 1,3,7;3\rangle$ is bipartite and, being hamiltonian, n must be even. □

Theorem 2.2. *$T_n\langle 1,3,7;5\rangle$ is hamiltonian if and only if n is even.*

Proof. Let n be even. Clearly $n \geq 8$.

If $n \cong 0\, mod\, 4$.

The smallest n is 8. A hamiltonian cycle in $T_8\langle 1,3,7;5\rangle$ is $(1,8,3,4,7,\ 2,5,6,1)$, which does not contain

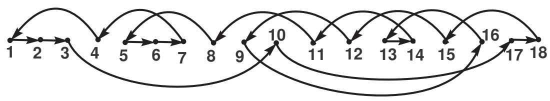

Figure 7: Hamiltonian cycle in $T_{18}\langle 1,3,7;3\rangle$

the edge $(6,7)$. The next representative in this class is 12, and a hamiltonian cycle in $T_{12}\langle 1,3,7;5\rangle$ is $(1,8,9,\underline{10,11},12,7,2,3,4,5,6,1)$.

If $n \cong 2\, mod\, 4$.
The smallest n is 10. A hamiltonian cycle in $T_{10}\langle 1,3,7;5\rangle$ is $(1,2,3,4,7,\underline{8,9},10,5,6,1)$.

Note that for $n = 10$ and 12, these hamiltonian cycles contain the edge $(n-2, n-1)$. Thus, by Remark 1, these hamiltonian cycles in $T_{n\in\{10,12\}}\langle 1,3,7;5\rangle$ can be transformed into hamiltonian cycles in $T_{n+4}\langle 1,3,7;5\rangle$ by replacing the edge $(n-2,n-1)$ with the path $(n-2,n+1,\underline{n+2,n+3},n+4,n-1)$, which preserves the same property. Suppose, for some non-negative integer r, $T_{n=n_o+4r}\langle 1,3,7;5\rangle$, where n is even and different from 12, has a hamiltonian cycle containing the edge $(n-2,n-1)$ then $T_{n+4}\langle 1,3,7;5\rangle$ enjoys the same property and thus $T_n\langle 1,3,7;5\rangle$ is hamiltonian for all even n.

Conversely, $T_n\langle 1,3,7;5\rangle$ is bipartite and, being hamiltonian, n must be even. □

Theorem 2.3. *$T_n\langle 1,3,7;7\rangle$ is hamiltonian if and only if n is even.*

Proof. Let n be even. Clearly $n \geq 8$.

If $n \cong 0\, mod\, 6$.
The smallest n is 12, and a hamiltonian cycle in $T_{12}\langle 1,3,7;7\rangle$ is $(1,2,9,10,3,4,11,12,\ 5,6,7,8,1)$, which does not contain the edge $(10,11)$. The next representative in this class is 18, and a hamiltonian cycle in $T_{18}\langle 1,3,7;7\rangle$ is $(1,4,7,14,15,18,11,12,13,\underline{16,17},10,3,6,\ 9,2,5,8,1)$.

If $n \cong 2\, mod\, 6$.
The smallest n is 8. A hamiltonian cycle in $T_8\langle 1,3,7;7\rangle$ is $(1,2,3,4,5,\underline{6,7},8,1)$.

If $n \cong 4\, mod\, 6$.
The smallest n is 10, and a hamiltonian cycle in $T_{10}\langle 1,3,7;7\rangle$ is $(1,2,9,10,3,4,5,6,7,\ 8,1)$, which does not contain the edge $(8,9)$. The next representative in this class is 16, and a hamiltonian cycle in $T_{16}\langle 1,3,7;7\rangle$ is $(1,4,5,12,13,16,9,2,3,6,7,10,11,\underline{14,15},8,1)$.

Note that for $n = 8, 16$ and 18, these hamiltonian cycles contain the edge $(n-2, n-1)$. Thus, by Remark 1, these hamiltonian cycles in $T_{n\in\{8,16,18\}}\langle 1,3,7;7\rangle$ can be transformed into hamiltonian cycles in $T_{n+6}\langle 1,3,7;7\rangle$ by replacing the edge $(n-2,n-1)$ with the path $(n-2,n+1,n+2,n+3,\underline{n+4,n+5},n+6,n-1)$, which preserves the same property. Thus $T_n\langle 1,3,7;7\rangle$ is hamiltonian for all even n.

Conversely, $T_n\langle 1,3,7;7\rangle$ is bipartite and, being hamiltonian, n must be even. □

Theorem 2.4. *For odd b, $9 \leq b \leq 15$, $T_n\langle 1,3,7;b\rangle$ is hamiltonian if and only if n is even.*

Proof. Let b be odd and $9 \leq b \leq 15$, and n be even. Clearly $n \geq b+1$.

Case 1. Let $n \cong 0\, mod\, (b-1)$. The smallest n is $2b-2$. For $b \in \{9,13\}$, hamiltonian cycles in $T_{n=2b-2}\langle 1,3,7;b\rangle$ is $(1,2,\ldots,b-4,b-1,b+2,b+3,b+4) \cup C_{b+4\to b+8} \cup C_{b+8\to b+12} \cup \cdots \cup C_{n-7\to n-3} \cup (n-3,n,n-b = b-2,b+5,b+6) \cup C_{b+6\to b+10} \cup C_{b+10\to b+14} \cup \cdots \cup C_{n-5\to n-1} \cup (n-1,n-b-1 = b-3,b,b+1,1)$, see Figure 8. For $b = 11$, a hamiltonian cycle in $T_{20}\langle 1,3,7;11\rangle$ is $(1,2,3,10,13,14,17,\underline{18,19},20,9,16,5,6,7,8,15,4,11,12,1)$. And for $b = 15$, a hamiltonian cycle in $T_{28}\langle 1,3,7;15\rangle$ is $(1,2,3,6,7,14,17,18,21,22,25,\underline{26,27},28,13,20,23,24,9,10,\ 11,12,19,4,5,8,15,16,1)$.

Case 2. Let $n \cong 2\, mod\, (b-1)$. The smallest n is $b+1$. A hamiltonian cycle in $T_{n=b+1}\langle 1,3,7;b\rangle$ is $(1,2,3,\ldots,\underline{n-2,n-1},n = b+1,1)$.

Case 3. Let $n \cong 4\, mod\, (b-1)$. The smallest n is $b+3$. For $b = 9$ and 13, hamiltonian cycles in $T_{12}\langle 1,3,7;9\rangle$ and $T_{16}\langle 1,3,7;13\rangle$ are $(1,4,7,8,11,2,5,12,3,6,9,10,1)$ and $(1,4,7,10,13,16,3,6,9,12,15,2,5,8,11,14,1)$,

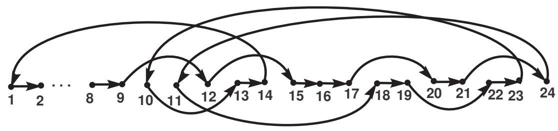

Figure 8: Hamiltonian cycle in $T_{24}\langle 1,3,7;13\rangle$

respectively. And for $b \in \{11,15\}$, hamiltonian cycles in $T_{n=b+3}\langle 1,3,7;b\rangle$ is $(1,2) \cup C_{2\to 6} \cup C_{6\to 10} \cup \cdots \cup C_{n-4\to n} \cup (n,3,4) \cup C_{4\to 8} \cup C_{8\to 12} \cup \cdots \cup C_{b-3\to b+1} \cup (b+1,1)$, see Figure 9. All these hamiltonian cycles in

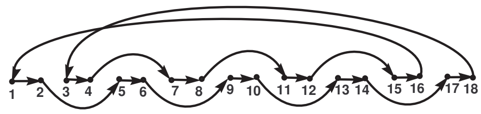

Figure 9: Hamiltonian cycle in $T_{18}\langle 1,3,7;15\rangle$

$T_{n=b+3}\langle 1,3,7;b\rangle$ do not contain the edge $(n-2,n-1)$. Now the next representative in this class is $n = 2b+2$. For $b \in \{9,13\}$, hamiltonian cycles in $T_{n=2b+2}\langle 1,3,7;b\rangle$ is $(1,2,\ldots,b-2,b+5,b+6) \cup C_{b+6\to b+10} \cup C_{b+10\to b+14} \cup \cdots \cup C_{n-5\to n-1} \cup (n-1,n,n-b=b+2,b+3,b+4) \cup C_{b+4\to b+8} \cup C_{b+8\to b+12} \cup \cdots \cup C_{n-7\to n-3} \cup (n-3,n-3-b=b-1,b,b+1,1)$, see Figure 10. And for $b \in \{11,15\}$, hamiltonian cycles in $T_{n=2b+2}\langle 1,3,7;b\rangle$ is $C_{1\to 5} \cup C_{5\to 9} \cup \cdots \cup C_{b-6\to b-2} \cup (b-2,b+5,b+6) \cup C_{b+6\to b+10} \cup C_{b+10\to b+14} \cup \cdots \cup C_{n-7\to n-3} \cup (n-3,n,n-b=b+2,2,3) \cup C_{3\to 7} \cup C_{7\to 10} \cup \cdots \cup C_{n-5\to n-1} \cup (n-1,n-b-1=b+1,1)$, see Figure 11.

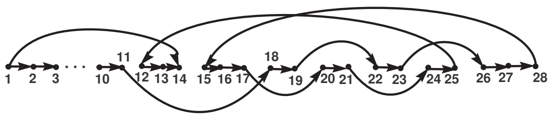

Figure 10: Hamiltonian cycle in $T_{28}\langle 1,3,7;13\rangle$

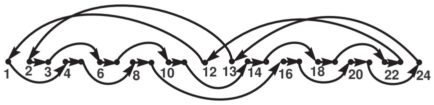

Figure 11: Hamiltonian cycle in $T_{24}\langle 1,3,7;11\rangle$

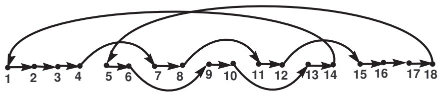

Figure 12: Hamiltonian cycle in $T_{18}\langle 1,3,7;13\rangle$

Case 4. Let $n \cong 6\, mod\, (b-1)$. The smallest n is $b+5$. For $b \in \{9,13\}$, a hamiltonian cycle in $T_{n=b+5}\langle 1,3,7;b\rangle$ is $(1,2,3,4) \cup C_{4\to 8} \cup C_{8\to 12} \cup \cdots \cup C_{b-6\to n-2} \cup (\underline{n-2,n-1}, n=b+5,5,6) \cup C_{6\to 10} \cup C_{10\to 14} \cup \cdots \cup C_{b-3\to b+1} \cup (b+1,1)$, see Figure 12. For $b = 11$ and 15, hamiltonian cycles in $T_{16}\langle 1,3,7;11\rangle$ and $T_{20}\langle 1,3,7;15\rangle$ are $(1,8,11,\underline{14,15},4,7,10,13,2,3,6,9,16,5,12,1)$ and $(1,8,15,\underline{18,19},4,7,14,17,2,3,10,11,12,13,20,5,6,9,16,1)$

Case 5. Let $n \cong 8\, mod\,(b-1)$. The smallest n is $b+7$. Clearly here $b > 9$. For $b = 13$, a hamiltonian cycle in $T_{20}\langle 1,3,7;13\rangle$ is $(1,2,3,4,5,12,15,18,19,6,13,16,17,20,7,8,9,10,11,\ 14,1)$. For $b \in \{11,15\}$, a hamiltonian cycle in $T_{n=b+7}\langle 1,3,7;b\rangle$ is $(1,2,\ldots,6) \cup C_{6\to 10} \cup C_{10\to 14} \cup \cdots \cup C_{n-8\to n-4} \cup (n-4, n-3, n-2, n-1, n = b+7, 7, 8) \cup C_{8\to 12} \cup C_{12\to 16} \cup \cdots \cup C_{b-3\to b+1} \cup (b+1,1)$, see Figure 13.

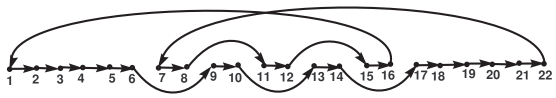

Figure 13: Hamiltonian cycle in $T_{22}\langle 1,3,7;15\rangle$

Case 6. Let $n \cong 10\, mod\,(b-1)$. The smallest n is $b+9$. Clearly here $b > 11$. For $b = 13$, a hamiltonian cycle in $T_{22}\langle 1,3,7;13\rangle$ is $(1,2,3,4,5,6,7,8,11,12,15,16,17,18,19,20,21,22,9,10,13,14,1)$. For $= 15$, a hamiltonian cycle in $T_{24}\langle 1,3,7;15\rangle$ is $(1,4,5,12,15,18,19,22,23,\ 8,11,14,17,\ 2,3,6,7,10,13,20,21,24,9,16,1)$.

Case 7. Let $n \cong 12\, mod\,(b-1)$. The smallest n is $b+11$. Clearly here $b > 13$. For $b = 15$, a hamiltonian cycle in $T_{26}\langle 1,3,7;15\rangle$ is $(1,2,\ldots,10,13,14,17,18,\ldots,24,25,26,11,12,\ 15,16,1)$.

All these hamiltonian cycles in each class (except $n = b+3$) contain the edge $(n-2, n-1)$. Suppose, for some non-negative integer r, $T_{n=n_o+r(b-1)}\langle 1,3,7;b\rangle$ has a hamiltonian cycle containing the edge $(n-2,n-1)$, where n is even and different from $b+3$, then, by Remark 1, $T_{n+b-1}\langle 1,3,7;b\rangle$ enjoys the same property. Thus $T_n\langle 1,3,7;b\rangle$ is hamiltonian for all even n, where b is odd and $9 \le b \le 15$.

Conversely, $T_n\langle 1,3,7;b\rangle$ is bipartite and, being hamiltonian, n must be even. $\square$

Theorem 2.5. *For odd $b > 15$, $T_n\langle 1,3,7;b\rangle$ is hamiltonian for all even n except ($n \cong 6,10,\ldots,b-3\, mod\,(b-1)$ and $b \cong 3\, mod\, 4$) or ($n \cong 8,12,\ldots,b-3\, mod\,(b-1)$ and $b \cong 1\, mod\, 4$).*

Proof. Let b be odd and greater that 15, and n be even.

Case 1. Let $n \cong 0\, mod\,(b-1)$. The smallest $n > b$, is $2b-2$. If $b \cong 1\, mod\, 4$, then a hamiltonian cycles in $T_{n=2b-2}\langle 1,3,7;b\rangle$ is $(1,2,\ldots,b-4,b,b+2,b+3,b+4) \cup C_{b+4\to b+8} \cup C_{b+8\to b+12} \cup \cdots \cup C_{n-7\to n-3} \cup (n-3, n, n-b = b-2, b+5, b+6) \cup C_{b+6\to b+10} \cup C_{b+10\to b+14} \cup \cdots \cup C_{n-5\to n-1} \cup (n-1, n-b-1 = b-3, b, b+1, 1)$, see Figure 14. If $b \cong 3\, mod\, 4$, then a hamiltonian cycle in $T_n\langle 1,3,7;b\rangle$ is $(1,2,3) \cup C_{3\to 7} \cup C_{7\to 11} \cup \cdots \cup C_{b-12\to b-8} \cup$

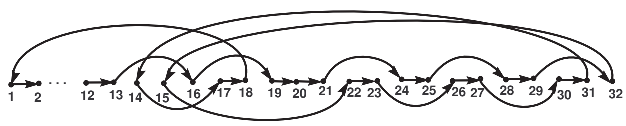

Figure 14: Hamiltonian cycle in $T_{32}\langle 1,3,7;17\rangle$

$(b-8, b-1) \cup C_{b-1\to b+3} \cup C_{b+3\to b+7} \cup \cdots \cup C_{n-6\to n-2} \cup (n-2, n-1, n, n-b = b-2, b+5) \cup C_{b+5\to b+9} \cup C_{b+9\to b+13} \cup \cdots \cup C_{n-8\to n-4} \cup (n-4, n-4-b = b-6, b-5, b-4, b-3, b+4, 4, 5) \cup C_{5\to 9} \cup C_{9\to 13} \cup \cdots \cup C_{b-14\to b-10} \cup (b-10, b-7, b, b+1, 1)$, see Figure 15.

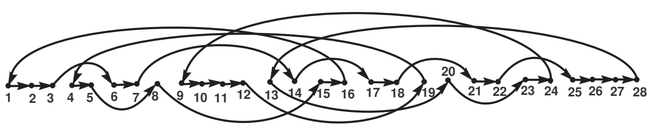

Figure 15: Hamiltonian cycle in $T_{28}\langle 1,3,7;15\rangle$

Case 2. Let $n \cong 2\, mod\,(b-1)$. The smallest n is $b+1$. A hamiltonian cycle in $T_{n=b+1}\langle 1,3,7;b\rangle$ is $(1,2,3,\ldots,n-2,n-1,n = b+1,1)$.

Case 3. Let $n \cong 4\,mod\,(b-1)$. The smallest n is $b+3$.
(i) Let $b \cong 1\,mod\,4$. If $n \cong 0\,mod\,3$, then a hamiltonian cycle in $T_{n=b+3}\langle 1,3,7;b\rangle$ is $(1,4,7,\ldots,n-5,n-4,n-1=b+2,2,5,8,\ldots,n-7,n=b+3,3,6,9,\ldots,n-3,n-2=b+1,1)$, see Figure 16.

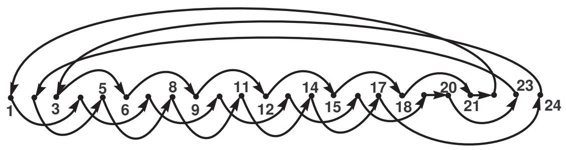

Figure 16: Hamiltonian cycles in $T_{24}\langle 1,3,7;21\rangle$

If $n \cong 1\,mod\,3$, then a hamiltonian cycle in $T_{n=b+3}\langle 1,3,7;b\rangle$ is $(1,4,7,\ldots,n,3,6,9,\ldots,n-1,2,5,8,\ldots,n-2,1)$, see Figure 17. If $n \cong 2\,mod\,3$, then a hamiltonian cycle in $T_{n=b+3}\langle 1,3,7;b\rangle$ is $(1,4,7,\ldots,n-1,2,5,8,\ldots,n,3,6,9,\ldots,n-2,1)$, see Figure 18.

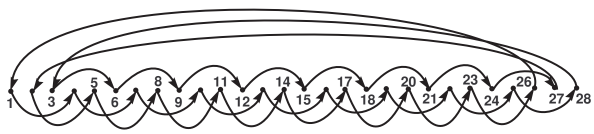

Figure 17: Hamiltonian cycles in $T_{28}\langle 1,3,7;25\rangle$

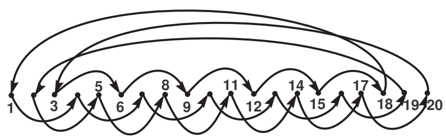

Figure 18: Hamiltonian cycles in $T_{20}\langle 1,3,7;17\rangle$

(ii) If $b \cong 3\,mod\,4$, then a hamiltonian cycle in $T_{n=b+3}\langle 1,3,7;b\rangle$ is $(1,2)\cup C_{2\to 6}\cup C_{6\to 10}\cup\cdots\cup C_{n-4\to n}\cup(n,3,4)\cup C_{4\to 8}\cup C_{8\to 12}\cup\cdots\cup C_{b-3\to b+1}\cup(b+1,1)$, see Figure 19.

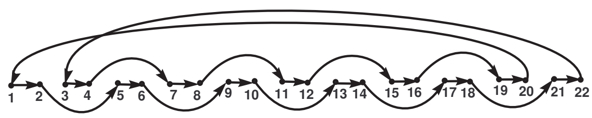

Figure 19: Hamiltonian cycles in $T_{22}\langle 1,3,7;19\rangle$

These hamiltonian cycles in $T_{n=b+3}\langle 1,3,7;b\rangle$ do not contain the edge $(n-2,n-1)$. Now the next representative in this class is $n=2b+2$. If $b \cong 1\,mod\,4$, then a hamiltonian cycles in $T_{n=2b+2}\langle 1,3,7;b\rangle$ is $(1,2,\ldots,b-2,b+5,b+6)\cup C_{b+6\to b+10}\cup C_{b+10\to b+14}\cup\cdots\cup C_{n-5\to n-1}\cup(n-1,n,n-b=b+2,b+3,b+4)\cup C_{b+4\to b+8}\cup C_{b+8\to b+12}\cup\cdots\cup C_{n-7\to n-3}\cup(n-3,n-3-b=b-1,b,b+1,1)$, see Figure 20. If $b \cong 3\,mod\,4$, then a hamiltonian cycles in $T_{n=2b+2}\langle 1,3,7;b\rangle$ is $T_{n=2b+2}\langle 1,3,7;b\rangle$ is $C_{1\to 5}\cup C_{5\to 9}\cup\cdots\cup C_{b-6\to b-2}\cup(b-2,b+5,b+6)\cup C_{b+6\to b+10}\cup C_{b+10\to b+14}\cup\cdots\cup C_{n-7\to n-3}\cup(n-3,n,n-b=b+2,2,3)\cup C_{3\to 7}\cup C_{7\to 10}\cup\cdots\cup C_{n-5\to n-1}\cup(n-1,n-b-1=b+1,1)$, see Figure 21.

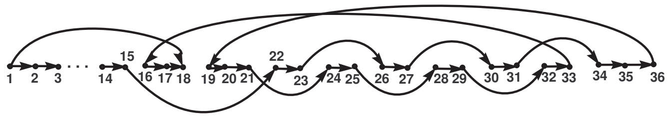

Figure 20: Hamiltonian cycles in $T_{36}\langle 1,3,7;17\rangle$

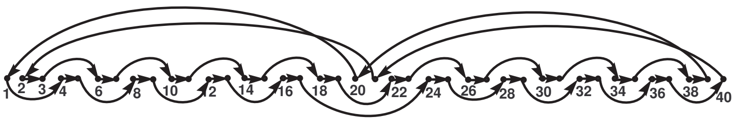

Figure 21: Hamiltonian cycles in $T_{40}\langle 1,3,7;19\rangle$

Case 4. Let $n \cong s \, mod\,(b-1)$, where $s \in \{6,10,\ldots,b-3\}$, and $b \cong 1\, mod\, 4$. The smallest n is $s+b-1$. Then a hamiltonian cycle in $T_{n=s+b-1}\langle 1,3,7;b\rangle$ is $(1,2,\ldots,s-2) \cup C_{s-2\to s+2} \cup C_{s+2\to s+6} \cup \cdots \cup C_{b-1\to b+3} \cup (b+3,b+4,\ldots,\underline{n-2,n-1},n,n-b=s-1,s) \cup C_{s\to s+4} \cup C_{s+4\to s+8} \cup \cdots \cup C_{b-3\to b+1} \cup (b+1,1)$, see Figure 22.

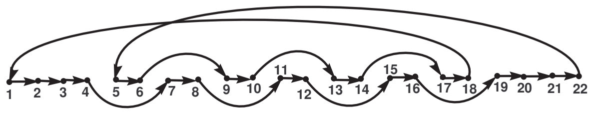

Figure 22: Hamiltonian cycles in $T_{22}\langle 1,3,7;17\rangle$ with $s=6$

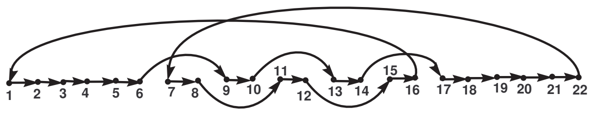

Figure 23: Hamiltonian cycles in $T_{22}\langle 1,3,7;15\rangle$ with $r=8$

Case 5. Let $n \cong r \, mod\,(b-1)$, where $r \in \{8,12,\ldots,b-3\}$ and $b \cong 3\, mod\, 4$. The smallest n is $r+b-1$. Then a hamiltonian cycle in $T_{n=r+b-1}\langle 1,3,7;b\rangle$ is $(1,2,\ldots,r-2) \cup C_{r-2\to r+2} \cup C_{r+2\to r+6} \cup \cdots \cup C_{b-1\to b+3} \cup (b+3,b+4,\ldots,\underline{n-2,n-1},n,n-b=r-1,r) \cup C_{r\to r+4} \cup C_{r+4\to r+8} \cup \cdots \cup C_{b-3\to b+1} \cup (b+1,1)$, see Figure 23.

All these hamiltonian cycles (except for $n=b+3$) contain the edge $(n-2,n-1)$, by using the technique of Remark 1, these hamiltonian cycles in $T_n\langle 1,3,7;b\rangle$, can be transformed into those in $T_{n+b-1}\langle 1,3,7;b\rangle$ which enjoys the same property. Thus $T_n\langle 1,3,7;b\rangle$ is hamiltonian for all even n except ($n \cong 6,10,\ldots,b-3\, mod\,(b-1)$ and $b \cong 3\, mod\, 4$) or ($n \cong 8,12,\ldots,b-3\, mod\,(b-1)$ and $b \cong 1\, mod\, 4$). □

Conjecture: For odd $b>15$, $T_n\langle 1,3,7;b\rangle$ is hamiltonian for all even n and ($n \cong 6,10,\ldots,b-3\, mod\,(b-1)$ and $b \cong 3\, mod\, 4$) or ($n \cong 8,12,\ldots,b-3\, mod\,(b-1)$ and $b \cong 1\, mod\, 4$).

For odd $b>1$, if $T_n\langle 1,3,7;b\rangle$ is hamiltonian, then n must be even because $T_n\langle 1,3,7;b\rangle$ is bipartite. Then the above conjecture, along with Theorem 2.5, will imply that for odd $b>1$, $T_n\langle 1,3,7;b\rangle$ is hamiltonian if and only if n is even.

3. Toeplitz Graphs $T_n\langle 1,3,7;b\rangle$ for even b

Now we will discus the hamiltonicity of $T_n\langle 1,3,7;b\rangle$ for even b. We will see that for $b<15$, only for fintie many exception of n, $T_n\langle 1,3,7;b\rangle$ is hamiltonian for all n.

Theorem 3.1. *$T_n\langle 1,3,7;2\rangle$ is hamiltonian for all n different from* $8,11,12,13,14,18,19,$ $20,25,26,32$.

Proof.

If $n\cong 0\,mod\,7$ and $n\neq 14$. Then a hamiltonian cycle in $T_n\langle 1,3,7;2\rangle$ is $(1,8,15,\ldots,n-6,n-8,n-1,n,n-2,n-4,n-3,n-5,n-7,n-9,n-11,n-10,n-12)\cup D_{n-12\to n-19}\cup D_{n-19\to n-26}\cup\cdots\cup D_{16\to 9}\cup(9,7,5,6,4,2,3,1)$, see Figure 24.

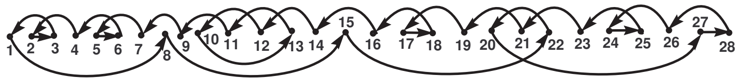

Figure 24: Hamiltonian cycles in $T_{28}\langle 1,3,7;2\rangle$

If $n\cong 1\,mod\,7$ and $n\neq 8$. Then a hamiltonian cycle in $T_n\langle 1,3,7;2\rangle$ is $(1,2,9,16,23,\ldots,n-6,n-4,n-1,n,n-2,n-4,n-3,n-5,n-7,n-9,n-11,n-10,n-12)\cup D_{n-12\to n-19}\cup D_{n-19\to n-26}\cup\cdots\cup D_{10\to 3}\cup(3,1)$, see Figure 25.

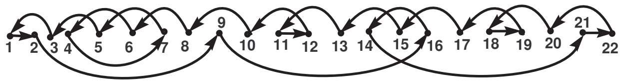

Figure 25: Hamiltonian cycles in $T_{22}\langle 1,3,7;2\rangle$

If $n\cong 2\,mod\,7$. Then a hamiltonian cycle in $T_{n\neq 9}\langle 1,3,7;2\rangle$ is $(1,8,15,\ldots,n-1,n-3,n,n-2,n-4,n-6,n-5,n-7)\cup D_{n-7\to n-14}\cup D_{n-14\to n-21}\cup\cdots\cup D_{16\to 9}\cup(9,7,5,6,4,2,3,1)$. And a hamiltonian cycle in $T_9\langle 1,3,7;2\rangle$ is $(1,8,6,4,2,9,7,5,3,1)$, see Figure 26.

If $n\cong 3\,mod\,7$. Then a hamiltonian cycle in $T_n\langle 1,3,7;2\rangle$ is $(1,2,9,16,\ldots,n-1,n-3,n,n-2,n-4,n-6,n-5,n-7)\cup D_{n-7\to n-14}\cup D_{n-14\to n-21}\cup\cdots\cup D_{10\to 3}\cup(3,1)$, see Figure 27.

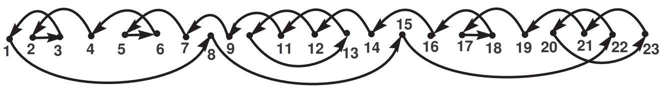

Figure 26: Hamiltonian cycles in $T_{23}\langle 1,3,7;2\rangle$

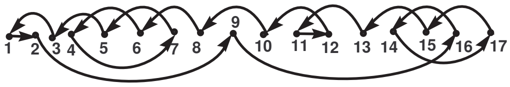

Figure 27: Hamiltonian cycles in $T_{17}\langle 1,3,7;2\rangle$

If $n\cong 4\,mod\,7$ and $n\notin\{11,18,25,32\}$. Clearly here $n\geq 39$. A hamiltonian cycle in $T_n\langle 1,3,7;2\rangle$ is $(1,2,9,16,23,\ldots,n-30,n-32,n-25,n-18,n-20,n-13,n-6,n-8,n-1,n,n-2,n-4,n-3,n-5,\ldots,n-11,n-10,n-12,n-14,n-16,n-15,n-17,n-19,n-21,n-23,n-22,n-24,n-26,n-28,n-27,n-29,n-31,n-33,n-35,n-34,n-36)\cup D_{n-36\to n-43}\cup D_{n-43\to n-50}\cup\cdots\cup D_{10\to 3}\cup(3,1)$.

If $n\cong 5\,mod\,7$ and $n\notin\{12,19,26\}$. Clearly here $n\geq 33$. A hamiltonian cycle in $T_n\langle 1,3,7;2\rangle$ is $(1,8,15,13,20,27,34,\ldots,n-6,n-4,n-1,n,n-2,n-4,n-3,n-5,n-7,n-9,n-11,n-10,n-12)\cup D_{n-12\to n-19}\cup D_{n-19\to n-26}\cup\cdots\cup D_{28\to 21}\cup(21,19,17,18,16,14,12,10,11,9,7,5,6,4,2,3,1)$.

If $n \cong 6\,mod\,7$ and $n \notin \{13, 20\}$. Clearly here $n \geq 27$. A hamiltonian cycle in $T_n\langle 1, 3, 7; 2\rangle$ is $(1, 8, 15, 13, 20, 27, 34, \ldots, n, n-2, n-1, n-3, n-5, n-4, n-6) \cup D_{n-6\to n-13} \cup D_{n-13\to n-20} \cup \cdots \cup D_{28\to 21} \cup (21, 19, 17, 18, 16, 14, 12, 10, 11, 9, 7, 5, 6, 4, 2, 3, 1)$.

This finishes the proof. □

Conjecture: $T_n\langle 1, 3, 7; 2\rangle$ is non hamiltonian for $n \in \{8, 11, 12, 13, 14, 18, 19, 20, 25, 26, 32\}$.

Theorem 3.2. *$T_n\langle 1, 3, 7; 4\rangle$ is hamiltonian for all n different from* 12.

Proof.

If $n \cong 0\,mod\,3$ and $n \neq 12$. The smallest n is 9, and the unique hamiltonian cycle in $T_9\langle 1, 3, 7; 4\rangle$ is $(1, 8, 4, 7, 3, 6, 2, 9, 5, 1)$ which does not contain the edge $(7, 8)$. Now the next representative in this class, different from 12, is 15, and a hamiltonian cycle in $T_{15}\langle 1, 3, 7; 4\rangle$ is $(1, 2, 3, 10, 6, 9, 12, \underline{13, 14}, 15, 11, 7, 8, 4, 5, 1)$.

If $n \cong 1\,mod\,3$. The smallest n is 10, and the unique hamiltonian cycle in $T_{10}\langle 1, 3, 7; 4\rangle$ is $(1, 8, 4, 7, 3, 10, 6, 2, 9, 5, 1)$ which does not contain the edge $(8, 9)$. Now the next representative in this class is 13, and a hamiltonian cycle in $T_{13}\langle 1, 3, 7; 4\rangle$ is $(1, 2, 3, 10, 6, 7, 8, 4, \underline{11, 12}, 13, 9, 5, 1)$.

If $n \cong 2\,mod\,3$. The smallest n is 8, and a hamiltonian cycle in $T_8\langle 1, 3, 7; 4\rangle$ is $(1, 2, 3, \underline{6, 7}, 8, 4, 5, 1)$.

These hamiltonian cycles for $n \in \{8, 13, 15\}$ contain the edge $(n-2, n-1)$, by using the technique of Remark 1, $T_{n+b-1}\langle 1, 3, 7; 4\rangle$ enjoys the same property. Thus $T_n\langle 1, 3, 7; 4\rangle$ is hamiltonian for all n different from 12.

This finishes the proof. □

Conjecture: $T_{12}\langle 1, 3, 7; 4\rangle$ is non hamiltonian.

Theorem 3.3. *$T_n\langle 1, 3, 7; 6\rangle$ is hamiltonian for all n.*

Proof. Clearly $n > 7$.

If $n \cong 0\,mod\,5$. The smallest n is 10, and a hamiltonian cycle in $T_{10}\langle 1, 3, 7; 6\rangle$ is $(1, 8, 2, 9, 3, 10, 4, 5, 6, 7, 1)$ which does not contain the edge $(8, 9)$. Now the next representative in this class is 15, and a hamiltonian cycle in $T_{15}\langle 1, 3, 7; 6\rangle$ is $(1, 2, 5, 6, \underline{13, 14}, 8, 11, 12, 15, 9, 3, 10, 4, 7, 1)$.

If $n \cong 1\,mod\,5$. The smallest n is 11, and a hamiltonian cycle in $T_{11}\langle 1, 3, 7; 6\rangle$ is $(1, 8, 2, 9, 3, 10, 4, 11, 5, 6, 7, 1)$ which does not contain the edge $(9, 10)$. A hamiltonian cycle in $T_{16}\langle 1, 3, 7; 6\rangle$ is $(1, 8, 2, 5, 12, 6, 9, 3, 4, 11, \underline{14, 15}, 16, 10, 13, 7, 1)$.

If $n \cong 2\,mod\,5$. The smallest n is 12, and a hamiltonian cycle in $T_{12}\langle 1, 3, 7; 6\rangle$ is $(1, 2, 3, 4, 5, 8, 9, \underline{10, 11}, 12, 6, 7, 1)$.

If $n \cong 3\,mod\,5$. The smallest n is 8, and a hamiltonian cycle in $T_8\langle 1, 3, 7; 6\rangle$ is $(1, 4, 5, 8, 2, 3, \underline{6, 7}, 1)$.

If $n \cong 4\,mod\,5$. The first and second smallest n are 9 and 14, and hamiltonian cycles in $T_9\langle 1, 3, 7; 6\rangle$ and $T_{14}\langle 1, 3, 7; 6\rangle$ are $(1, 8, 2, 9, 3, 4, 5, 6, 7, 1)$ and $(1, 4, 11, 14, 8, 2, 5, 12, 6, 9, 3, 10, 13, 7, 1)$, respectively. But, in both these cases, the edge $(n-2, n-1)$ is not in its hamiltonian cycle. A hamiltonian cycle in $T_{19}\langle 1, 3, 7; 6\rangle$ is $(1, 2, 3, 4, 5, 6, 9, 10, 11, 14, 8, 15, 16, \underline{17, 18}, 12, 19, 13, 7, 1)$.

For $n \in \{8, 12, 15, 16, 19\}$, these hamiltonian cycles in $T_n\langle 1, 3, 7; 6\rangle$ contain the edge $(n-2, n-1)$. By using the technique of Remark 1, these hamiltonian cycles can be transformed into those in $T_{n+b-1}\langle 1, 3, 7; 6\rangle$ which enjoys the same property. Thus $T_n\langle 1, 3, 7; 6\rangle$ is hamiltonian for all n.

This finishes the proof. □

Theorem 3.4. *For $b \in \{8, 10\}$, $T_n\langle 1, 3, 7; b\rangle$ is hamiltonian for all n different from $2b-6$.*

Proof. Let $b \in \{8, 10\}$. We need to show that $T_n\langle 1, 3, 7; 8\rangle$ and $T_n\langle 1, 3, 7; 10\rangle$ are hamiltonian for all n, differenet from 10 and 14, respectively.

If $n \cong 0\,mod\,b-1$. The smallest n is $2b-2$. Hamiltonian cycles in $T_{14}\langle 1, 3, 7; 8\rangle$ and $T_{18}\langle 1, 3, 7; 10\rangle$ are $(1, 8, 11, 3, 4, 5, 12, 13, 14, \underline{6, 7}, 10, 2, 9, 1)$ and $(1, 2, 3, 6, 13, \underline{16, 17}, 7, 14, 4, 5, 12, 15, 18, 8, 9, 10, 11, 1)$, respectively.

If $n \cong 1\, mod\, b-1$. The smallest n is $2b-1$. Hamiltonian cycles in $T_{15}\langle 1,3,7;8\rangle$ and $T_{19}\langle 1,3,7;10\rangle$ are $(1,2,3,10,11,12,4,5,6,\underline{13,14},15,7,8,9,1)$ and $(1,4,7,14,\underline{17,18},8,15,5,12,2,3,6,13,16,19,9,10,11,1)$, respectively.

If $n \cong 2\, mod\, b-1$. The smallest n is $b+1$, and a hamiltonian cycle in $T_{n=b+1}\langle 1,3,7;b\rangle$ is $(1,2,3,\dots,\underline{n-2,n-1},n,1)$.

If $n \cong 3\, mod\, b-1$. The smallest n is $b+2$. For $b=8$, the smallest n, different from 10, is 17, and a hamiltonian cycle in $T_{17}\langle 1,3,7;8\rangle$ is $(1,4,7,10,2,5,12,\underline{15,16},8,11,3,6,13,14,17,9,1)$. For $b=10$, a hamiltonian cycle in $T_{12}\langle 1,3,7;10\rangle$ is $(1,4,5,8,9,12,2,3,6,7,\underline{10,11},1)$.

If $n \cong 4\, mod\, b-1$. The smallest n is $b+3$. Hamiltonian cycles in $T_{11}\langle 1,3,7;8\rangle$ and $T_{13}\langle 1,3,7;10\rangle$ are $(1,8,11,3,4,5,6,7,10,2,9,1)$ and $(1,8,9,12,2,5,6,13,3,4,7,10,11,1)$, respectively. These hamiltonian cycles in $T_{b+3}\langle 1,3,7;b\rangle$ do not contain the edge $(n-2,n-1)$. Now the next representative in this class is $n=2b+2$, and hamiltonian cycles in $T_{18}\langle 1,3,7;8\rangle$ and $T_{22}\langle 1,3,7;10\rangle$ are $(1,8,11,18,10,2,3,4,5,12,15,7,14,6,13,\underline{16,17},9,1)$ and $(1,2,3,10,17,18,19,9,16,6,7,14,4,5,8,15,22,12,13,\underline{20,21},11,1)$, respectively.

If $n \cong 5\, mod\, b-1$. The smallest n is $b+4$. For $b=8$, a hamiltonian cycle in $T_{12}\langle 1,3,7;8\rangle$ is $(1,2,3,6,7,\underline{10,11},12,4,5,8,9,1)$. For $b=10$, the smallest n, different from 14, is 23, and a hamiltonian cycle in $T_{23}\langle 1,3,7;10\rangle$ is $(1,8,15,16,23,13,3,4,5,6,7,14,17,18,19,20,\underline{21,22},12,2,9,10,11,1)$.

If $n \cong 6\, mod\, b-1$. The smallest n is $b+5$. Hamiltonian cycles in $T_{13}\langle 1,3,7;8\rangle$ and $T_{15}\langle 1,3,7;10\rangle$ are $(1,8,11,3,6,13,5,12,4,7,10,2,9,1)$ and $(1,8,15,5,12,2,9,10,13,3,6,7,14,4,11,1)$, respectively. These hamiltonian cycles in $T_{b+5}\langle 1,3,7;b\rangle$ do not contain the edge $(n-2,n-1)$. Now, hamiltonian cycles in $T_{20}\langle 1,3,7;6\rangle$ and $T_{24}\langle 1,3,7;10\rangle$ are $(1,2,3,10,11,\underline{18,19},20,12,4,5,6,13,16,8,15,7,14,17,9,1)$ and $(1,2,3,6,9,16,17,7,8,15,18,21,24,14,4,5,12,19,\underline{22,23},13,20,10,11,1)$, respectively.

If $n \cong 7\, mod\, b-1$. Clearly here $b>8$, so we consider only $b=10$. The smallest n is 16, and a hamiltonian cycle in $T_{16}\langle 1,3,7;10\rangle$ is $(1,2,3,4,5,8,9,12,13,\underline{14,15},16,6,7,10,11,1)$.

If $n \cong 8\, mod\, b-1$. Clearly here $b>9$, so we consider only $b=10$. The smallest n is 17, and a hamiltonian cycle in $T_{17}\langle 1,3,7;10\rangle$ is $(1,2,3,4,5,12,\underline{15,16},6,13,14,17,7,8,9,10,11,1)$.

By using the technique of Remark 1, these hamiltonian cycles in $T_n\langle 1,3,7;b\rangle$ containing the edge $(n-2,n-1)$, can be transformed into those in $T_{n+b-1}\langle 1,3,7;b\rangle$ which enjoys the same property. Thus $T_n\langle 1,3,7;b\rangle$ is hamiltonian for all n different from $2b-6$.

This finishes the proof. □

Conjecture: $T_{10}\langle 1,3,7;8\rangle$ and $T_{14}\langle 1,3,7;10\rangle$ are non hamiltonian.

Theorem 3.5. *$T_n\langle 1,3,7;12\rangle$ is hamiltonian for all n different from 14 and 18.*

Proof. Clearly $n>13$.

If $n \cong 0\, mod\, 11$. The smallest n is 22, and a hamiltonian cycle in $T_{22}\langle 1,3,7;12\rangle$ is $(1,4,5,12,15,18,19,7,8,9,16,17,\underline{20,21},22,10,11,14,2,3,6,13,1)$.

If $n \cong 1\, mod\, 11$. The smallest n is 23, and a hamiltonian cycle in $T_{23}\langle 1,3,7;12\rangle$ is $(1,4,5,8,9,12,15,16,\dots,\underline{21,22},23,11,14,2,3,6,7,10,13,1)$.

If $n \cong 2\, mod\, 11$. The smallest n is 13, and a hamiltonian cycle in $T_{13}\langle 1,3,7;12\rangle$ is $(1,2,3,\dots,\underline{11,12},13,1)$.

If $n \cong 3\, mod\, 11$. The smallest n, different from 14, is 25, and a hamiltonian cycle in $T_{25}\langle 1,3,7;12\rangle$ is $(1,2,9,16,19,20,\underline{23,24},12,15,3,4,5,6,7,8,11,14,21,22,10,17,18,25,13,1)$.

If $n \cong 4\, mod\, 11$. The smallest n is 15, and a hamiltonian cycle in $T_{15}\langle 1,3,7;12\rangle$ is $(1,8,15,3,4,5,6,7,10,11,14,2,9,12,13,1)$ which does not contain the edge $(13,14)$. Now, a hamiltonian cycle in $T_{26}\langle 1,3,7;12\rangle$ is $(1,4,5,12,15,22,23,11,18,6,7,8,9,16,19,26,14,2,3,10,17,20,21,\underline{24,25},13,1)$.

If $n \cong 5\, mod\, 11$. The smallest n is 16, and a hamiltonian cycle in $T_{16}\langle 1,3,7;12\rangle$ is $(1,2,3,6,7,10,11,\underline{14,15},16,4,5,8,9,12,13,1)$.

If $n \cong 6\, mod\, 11$. The smallest n is 17, and a hamiltonian cycle in $T_{17}\langle 1,3,7;12\rangle$ is $(1,8,15,3,10,17,5,6,7,14,2,9,16,4,11,12,13,1)$ which does not contain the edge $(15,16)$. Now, a hamiltonian cycle in $T_{28}\langle 1,3,7;12\rangle$ is $(1,2,3,4,5,8,15,22,10,11,18,6,7,14,17,20,21,9,12,19,\underline{26,27},28,16,23,24,25,13,1)$.

If $n \cong 7 \, mod \, 11$. The smallest n, different from 18, is 29, and a hamiltonian cycle in $T_{29}\langle 1,3,7;12\rangle$ is $(1,2,3,4,11,12,15,18,19,20,\underline{27,28},16,23,24,25,26,14,21,22,29,17,5,6,7,8,9,10,13,1)$.

If $n \cong 8 \, mod \, 11$. The smallest n, is 19, and a hamiltonian cycle in $T_{19}\langle 1,3,7;12\rangle$ is $(1,4,5,8,11,12,15,16,\underline{17,18},19,7,14,2,3,6,9,10,13,1)$.

If $n \cong 9 \, mod \, 11$. The smallest n, is 20, and a hamiltonian cycle in $T_{20}\langle 1,3,7;12\rangle$ is $(1,2,3,4,5,6,7,10,11,14,15,16,17,\underline{18,19},20,8,9,12,13,1)$.

If $n \cong 10 \, mod \, 11$. The smallest n, is 21, and a hamiltonian cycle in $T_{20}\langle 1,3,7;12\rangle$ is $(1,2,3,6,7,14,17,5,12,\underline{19,20},8,15,16,4,11,18,21,9,10,13,1)$.

By using the technique of Remark 1, these hamiltonian cycles in $T_n\langle 1,3,7;12\rangle$ containing the edge $(n-2,n-1)$, can be transformed into those in $T_{n+b-1}\langle 1,3,7;b\rangle$ which enjoys the same property. Thus $T_n\langle 1,3,7;b\rangle$ is hamiltonian for all n different from 14 and 18.

This finishes the proof. □

Conjecture: $T_{n\in\{14,18\}}\langle 1,3,7;12\rangle$ is non hamiltonian.

Theorem 3.6. *$T_n\langle 1,3,7;14\rangle$ is hamiltonian for all n different from* 19.

Proof. Clearly $n > 14$.

If $n \cong 0 \, mod \, 13$. The smallest n is 26, and a hamiltonian cycle in $T_{26}\langle 1,3,7;14\rangle$ is $(1,2,3,6,7,14,17,20,21,\underline{24,25},26,12,13,16,19,22,23,9,10,11,18,4,5,8,15,1)$.

If $n \cong 1 \, mod \, 13$. The smallest n is 27, and a hamiltonian cycle in $T_{27}\langle 1,3,7;14\rangle$ is $(1,2,3,4,7,14,17,18,21,22,\underline{25,26},27,13,20,6,9,16,23,24,10,11,12,19,5,8,15,1)$.

If $n \cong 2 \, mod \, 13$. The smallest n is 15, and a hamiltonian cycle in $T_{15}\langle 1,3,7;14\rangle$ is $(1,2,3,\ldots,\underline{13,14},15,1)$.

If $n \cong s \, mod \, 13$, where $s \in \{3,7,11\}$. The smallest n is $s+13$. A hamiltonian cycle in $T_{n=s+13}\langle 1,3,7;14\rangle$ is $(1,2,\ldots,s-2) \cup C_{s-2\to s+2} \cup C_{s+2\to s+6} \cup \cdots \cup C_{1\to 13} \cup (13,16,17,18,\ldots,\underline{n-2,n-1},n,s-1,s) \cup C_{s\to s+4} \cup C_{s+4\to s+8} \cup \cdots \cup C_{b-3\to b+1} \cup (b+1,1)$.

If $n \cong 4 \, mod \, 13$. The smallest n is 17, and a hamiltonian cycle in $T_{17}\langle 1,3,7;14\rangle$ is $(1,8,11,14,3,4,5,6,7,10,13,16,2,9,12,15,1)$ which does not contain the edge $(15,16)$. Now, a hamiltonian cycle in $T_{30}\langle 1,3,7;14\rangle$ is $(1,2,3,4,11,18,25,26,12,13,20,6,7,14,21,22,23,9,10,17,24,27,\underline{28,29},30,16,19,5,8,15,1)$.

If $n \cong 5 \, mod \, 13$. The smallest n is 18, and a hamiltonian cycle in $T_{18}\langle 1,3,7;14\rangle$ is $(1,8,11,18,4,5,6,7,14,17,3,10,13,16,2,9,12,15,1)$ which does not contain the edge $(16,17)$. Now, a hamiltonian cycle in $T_{31}\langle 1,3,7;14\rangle$ is $(1,2,9,12,19,20,27,28,31,17,3,4,5,6,7,8,11,18,21,22,25,26,\underline{29,30},16,23,24,10,13,14,15,1)$.

If $n \cong 6 \, mod \, 13$. The smallest n, different from 19, is 32, and a hamiltonian cycle in $T_{32}\langle 1,3,7;14\rangle$ is $(1,2,3,10,17,24,25,11,12,13,20,6,7,14,21,22,23,9,16,19,26,27,28,29,\underline{30,31},32,18,4,5,8,15,1)$.

If $n \cong 8 \, mod \, 13$. The smallest n is 21, and a hamiltonian cycle in $T_{21}\langle 1,3,7;14\rangle$ is $(1,2,3,4,5,12,\underline{19,20},6,13,16,17,18,21,7,8,9,10,11,14,15,1)$.

If $n \cong 9 \, mod \, 13$. The smallest n is 22, and a hamiltonian cycle in $T_{22}\langle 1,3,7;14\rangle$ is $(1,4,5,6,13,\underline{20,21},7,14,17,18,19,22,8,9,16,2,3,10,11,12,15,1)$.

If $n \cong 10 \, mod \, 13$. The smallest n is 23, and a hamiltonian cycle in $T_{23}\langle 1,3,7;14\rangle$ is $(1,4,7,14,17,3,6,13,16,2,5,12,19,20,23,9,10,11,18,\underline{21,22},8,15,1)$.

If $n \cong 12 \, mod \, 13$. The smallest n is 25, and a hamiltonian cycle in $T_{25}\langle 1,3,7;14\rangle$ is $(1,2,3,4,5,6,7,8,9,16,19,20,\underline{23,24},10,17,18,21,22,25,11,12,13,14,15,1)$.

By using the technique of Remark 1, these hamiltonian cycles in $T_n\langle 1,3,7;14\rangle$ containing the edge $(n-2,n-1)$, can be transformed into those in $T_{n+b-1}\langle 1,3,7;14\rangle$ which enjoys the same property. Thus $T_n\langle 1,3,7;14\rangle$ is hamiltonian for all n different from 19.

This finishes the proof. □

Conjecture: $T_{19}\langle 1,3,7;14\rangle$ is non hamiltonian.

Conjecture: For even $b \geq 16$.

(a) If $b \cong 0 \, mod \, 4$, then $T_n\langle 1,3,7;b\rangle$ is hamiltonian for all n different from $b+2$.

(b) If $b \cong 2\, mod\, 4$, then $T_n\langle 1,3,7;b\rangle$ is hamiltonian for all n.

Concluding Remark: In literature the hamiltonicity of Toeplitz graphs $T_n\langle 1,3,a_3;b\rangle$, upto $a_3 = 6$, has been investigated. In this paper we investigated it for $a_3 = 7$, that is, we investigated the hamiltonicity in Toeplitz graphs $T_n\langle 1,3,7;b\rangle$ and stated some conjecture regarding this. The next task in our opinion is to complete the hamiltonicity investigation in Toeplitz graphs $T_n\langle 1,3,7,a_4,a_5,\ldots,a_p;b_1,b_2,\ldots,b_q\rangle$ by solving the stated conjectures in this paper.

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