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Journal of Prime Research in Mathematics, 20(2) (2024), 117–124

Journal of Prime Research in Mathematics

Insights into dual Rickart modules: Unveiling the role of second cosingular submodules

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Abstract

In this paper, we propose a new type of module by focusing on the second cosingular submodule of a module. We define a module M as weak T-dual Rickart if, for any homomorphism $\varphi \in End_R(M)$, the submodule $\varphi(\overline{Z}^2(M))$ lies above a direct summand of M. We prove that this property is inherited by direct summands of M. We also introduce weak T-dual Baer modules and provide a complete characterization of such modules where the second cosingular submodule is a direct summand. Furthermore, we present a characterization of (semi)perfect rings in which every (finitely generated) module is weak T-dual Rickart.

Keywords: dual Rickart module, T-dual Rickart module, wTd-Rickart module, t-dual Baer module, weak T-dual Baer module.
2010 MSC: 16D10, 16D40, 16D80.

1. Introduction

All rings considered in this paper will be associative with an identity element and all modules will be unitary right modules unless otherwise stated. Let R be a ring and M an R-module. $S = End_R(M)$ will denote the ring of all R-endomorphisms of M. We will use the notation $N \ll M$ to indicate that N is small in M (i.e. $L \neq M, L + N \neq M$). A small module is module which is a small submodule of another module. A submodule L of M is *essential in* M denoted by $L \leq_e M$, if for every nonzero submodule K of M, $L \cap K \neq 0$. A module is uniform if each nonzero submodule is essential in it. Dually, a module is hollow in case each proper submodule is small in that module. The notation $N \leq^{\oplus} M$ means that N is a direct summand of M, while $N \trianglerighteq M$ means that N is a fully invariant submodule of M, i.e., every endomorphism of M maps N to a submodule in N. The terms radical and socle refer to specific submodules of a given module.

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Received : 01 August 2023*; Accepted:* 31 December 2024*; Published Online:* 31 December 2024.

Let $L \leq K \leq M$. We say that K lies above L in M if $K/L \ll M/L$. A module M is called lifting if every submodule H of M lies above a direct summand D of M ([5]). We may recall the definition of a supplemented modules. A module M is said to be supplemented provided for each submodule A in M, there exists a submodule B in M with $M = A + B$ and $A \cap B \ll B$. A strange variation of supplemented modules is the class of H-supplemented modules. A module M is called H-supplemented in case for each $N \leq M$, there exists a direct summand D of M such that $M = N + X$ if and only if $M = D + X$, for all $X \leq M$. All these notions and their relations can be found in [10] and [5]. It is important to state that, any lifting module is H-supplemented and each H-supplemented module is supplemented. Also, there are some works and studies about H-supplemented modules in the literature such as [16], [14] and [11, 12].

The study of Rickart and Baer modules has its roots in functional analysis with close links to C*-algebra and Von Neumann algebras. The concept of (dual) Rickart modules, where the (image) kernel of every endomorphism is a direct summand of the module, is important in the study of modules. Understanding idempotents in the endomorphism rings is also crucial ([9]). A module is Rickart and dual Rickart if and only if its endomorphism ring is von Neumann regular. A generalization of lifting modules and dual Rickart modules are $\mathcal{I}$-lifting modules, which have been characterized as direct sums of cyclic modules ([1]). These modules are more challenging to study compared to dual Rickart modules, but they have potential for further investigation. Amouzegar also presents a way to describe $\mathcal{I}$-lifting rings through finitely supplemented modules.

In [3], the authors introduced various types of $\mathcal{I}$-lifting modules based on a fixed fully invariant submodule in the given module. A module M is said to be $\mathcal{I}_F$-lifting, where F is a fully invariant submodule of M, if the submodule $\phi(F)$ lies above a direct summand D of M for every endomorphism ϕ of M. It is noted that a module M is $\mathcal{I}$-lifting if and only if it is $\mathcal{I}_M$-lifting. The properties of such modules are discussed in detail in [3]. Continuing this line of work, Moniri and Amouzegar in [13] investigated H-supplemented modules using the same approach as [3]. A module M is called $\mathcal{I}_F$-H-supplemented if, for every $\phi \in End_R(M)$, there exists a direct summand D of M such that $\phi(F) + X = M$ if and only if $D + X = M$, for all submodules X of M. The authors present certain conditions to ensure that a $\mathcal{I}_F$-H-supplemented module is $\mathcal{I}_F$-lifting. They also investigate the relationship between these and other similar classes of modules, and consider direct sums of $\mathcal{I}_F$-H-supplemented modules.

The authors in [4] examined dual Rickart modules using preradicals and provided a way to identify these modules based on a particular preradical. It is important to mention that any submodule which is fully invariant within a module can create a preradical.

The singular submodule $Z(M)$ of a module M is the set of $m \in M$ such that, $mI = 0$ for some essential right ideal I of R. Let M be an R-module. In [17], the authors defined $\overline{Z}(M)$ as a dual of singular submodule as follows: $\overline{Z}(M) = \bigcap\{Kerf \mid f\colon M \to U, U \in \mathcal{S}\}$ where $\mathcal{S}$ denotes the class of all small modules. They called M a *cosingular* (*noncosingular*) module if $\overline{Z}(M) = 0$ ($\overline{Z}(M) = M$). Clearly every small module is cosingular.

$Rad(M)$, $Soc(M)$ and $E(M)$ denote the radical, the socle and the injective envelope of a module M, respectively, and $J(R)$ denotes the Jacobson radical of a ring R.

In [18], Tribak introduced and investigated the notion called wd-Rickart modules, which is a generalization of the concept of d-Rickart modules. A module M is said to be wd-Rickart(weak dual Rickart) if for every nonzero endomorphism ϕ of M, $\phi(M)$ contains a nonzero direct summand of M.

According to previous research, this work can be motivated by the desire to further explore and expand on the recent works on dual Rickart modules and their generalizations. By building on the foundation laid by the authors in [4], [3], and [13], we aim to contribute to the existing body of knowledge in this field. By the way, we are interested in studying a special kind of $\mathcal{I}_F$-lifting modules where we shall replace the fully invariant submodule F by $\overline{Z}^2(M)$. In this regard, we produce a new class of modules namely wTd-Rickart modules. We call a module M, weak T-dual Rickart (wTd-Rickart) provided for each endomorphism f of M, there is a direct summand D of M such that $f(\overline{Z}^2(M))/D \ll M/D$. Some general properties of such modules are also considered. We prove that for a noncosingular module, the two concepts wTd-Rickart and $\mathcal{I}$-lifting coincide. It is also shown that every direct summand of any wTd-Rickart module inherits the property. We also introduce weak T-dual Baer modules.

2. Weak T-dual Rickart Modules

The authors in [6] introduced a new concept called T-dual Rickart modules which is a generalization of dual Rickart modules. A module M is considered T-dual Rickart if for every endomorphism φ of M, the image of the second cosingular submodule under φ is a direct summand of M. The second cosingular submodule is an important submodule of M and is used to study dual Rickart modules. A natural question that arises is what happens if we extend the concept of T-dual Rickart to its lifting version.

Definition 2.1. Let M be a module. Then we say M is wTd-Rickart, in case for every $\varphi \in S$, there exists a direct summand D of M contained in $\varphi(\overline{Z}^2(M))$ such that $\frac{\varphi(\overline{Z}^2(M))}{D} \ll \frac{M}{D}$.

Note that each T-dual Rickart module is wTd-Rickart. By the definitions, lifting modules provide a enormous source of wTd-Rickart modules. Although, there are some wTd-Rickart non-lifting modules such as $\mathbb{Z}$ as an $\mathbb{Z}$-module. In general, any cosingular non-lifting module is wTd-Rickart. The following presents a characterization of a wTd-Rickart module M with $\overline{Z}^2(M)$ a direct summand of M.

By the definitions, it can easily seen that wTd-Rickart modules generalize both lifting modules and dual Rickart modules. The following example includes this fact that each module can construct a wTd-Rickart module.

Example 2.2. (1) Every cosingular module is wTd-Rickart, so that every small module is too. In particular for every module M, the module $\frac{M}{\overline{Z}(M)}$ is wTd-Rickart. Also as an $\mathbb{Z}$-module for every $n \in \mathbb{N}$, the module $\mathbb{Z}_n$ is wTd-Rickart.

(2) Every T-dual Rickart is wTd-Rickart. Consider the $\mathbb{Z}$-module $M = \mathbb{Z}_{p^\infty} \oplus \mathbb{Z}_p$. By [6, Example 3.5], M is T-dual Rickart and so is wTd-Rickart. Note that by [9, Example 2.10], M is not dual Rickart.

(3) The ring $R = \prod_{i=1}^{\infty} \mathbb{Z}_2$ is a regular ring. Therefore R is dual Rickart module. Since R is V-ring, so by [6] R is a T-dual Rickart R-module, hence is wTd-Rickart.

The following gives us a characterization of wTd-Rickart modules with second cosingular submodule a direct summand.

Theorem 2.3. *Let M be a module such that $\overline{Z}^2(M)$ is a direct summand of M. Then M is wTd-Rickart if and only if $\overline{Z}^2(M)$ is $\mathcal{I}$-lifting.*

Proof. ($\Rightarrow$) Let $g : \overline{Z}^2(M) \to \overline{Z}^2(M)$ be an endomorphism of $\overline{Z}^2(M)$ and $\overline{Z}^2(M) \oplus L = M$ for a submodule L of M. Then $h = j \circ g \circ \pi_{\overline{Z}^2(M)} : M \to M$ is an endomorphism of M where $j : \overline{Z}^2(M) \to \overline{Z}^2(M)$ is the inclusion and $\pi_{\overline{Z}^2(M)} : M \to \overline{Z}^2(M)$ is the projection map on $\overline{Z}^2(M)$. It is easy to check that $h(\overline{Z}^2(M)) = g(\overline{Z}^2(M))$. As M is wTd-Rickart, there exists a direct summand D of M such that $g(\overline{Z}^2(M))/D \ll M/D$ and hence $g(\overline{Z}^2(M))/D \ll \overline{Z}^2(M)/D$. It is left to reader to verify that $\overline{Z}^2(M)/D$ is a direct summand of M/D.

($\Leftarrow$) Let $\overline{Z}^2(M)$ be $\mathcal{I}$-lifting and f be an endomorphism of M. Consider $q = \pi_{\overline{Z}^2(M)} \circ f \circ j : \overline{Z}^2(M) \to \overline{Z}^2(M)$, which is an endomorphism of $\overline{Z}^2(M)$, where $j : \overline{Z}^2(M) \to M$ is the inclusion and $\pi_{\overline{Z}^2(M)} : M \to \overline{Z}^2(M)$ is the projection on $\overline{Z}^2(M)$. Being $\overline{Z}^2(M)$ a fully invariant submodule of M implies that $q(\overline{Z}^2(M)) = f(\overline{Z}^2(M))$. As $\overline{Z}^2(M)$ is $\mathcal{I}$-lifting, there is a direct summand D of $\overline{Z}^2(M)$ (so that of M) such that $q(\overline{Z}^2(M))/D = f(\overline{Z}^2(M))/D \ll \overline{Z}^2(M)/D$. Therefore, M is wTd-Rickart. □

We shall present a characterization of wTd-Rickart modules with no nonzero small submodules.

Theorem 2.4. *Let M be a module with $Rad(M) = 0$. Then the following statements are equivalent:*
(1) *M is wTd-Rickart;*
(2) *$\overline{Z}^2(M)$ is a dual Rickart direct summand of M.*

Proof. (1) ⇒ (2) Let φ be an arbitrary endomorphism of M. Then there exists a direct summand D of M such that $\varphi(\overline{Z}^2(M))/D \ll M/D$. Since $Rad(M) = 0$ we conclude that $Rad(M/D) = 0$ as D is a direct summand of M. Therefore, $\varphi(\overline{Z}^2(M)) = D$ is a direct summand of M. It follows that $\overline{Z}^2(M)$ is a direct summand of M. From Theorem 2.3, $\overline{Z}^2(M)$ is $\mathcal{I}$-lifting. Since $Rad(M) = 0$, we conclude from [1, Proposition 3.1] that $\overline{Z}^2(M)$ is a dual Rickart module.

(2) ⇒ (1) It follows directly from Theorem 2.3 and the fact that every dual Rickart module is $\mathcal{I}$-lifting. □

From Theorem 2.4, we conclude that over a right V-ring R, a right R-module M is wTd-Rickart if and only if $\overline{Z}^2(M)$ is a dual Rickart direct summand of M.

To find a wTd-Rickart module which is not T-dual Rickart, we may consider a module M with $0 \neq \overline{Z}^2(M) \ll M$. Then it is clear that for every $f \in End(M)$, we have $f(\overline{Z}^2(M)) \ll M$. Since $\overline{Z}^2(M) \ll M$, it is clear that M can not be T-dual Rickart.

Proposition 2.5. *Let M be an indecomposable module. Then the following statements can be easily checked.*
(1) *M is T-dual Rickart if and only if $\varphi(\overline{Z}^2(M)) = 0$ or $\varphi(\overline{Z}^2(M)) = M$, for each $\varphi \in End_R(M)$.*
(2) *M is wTd-Rickart if and only if $\varphi(\overline{Z}^2(M)) \ll M$, for each $\varphi \in End_R(M)$.*

We next provide a condition for a module to be wTd-Rickart according cyclic ideals of $End_R(M)$.

Proposition 2.6. *Let M be a module. Then M is wTd-Rickart if and only if for every cyclic right ideal I of $S = End_R(M)$, the submodule $\sum_{\phi\in I}\phi(\overline{Z}^2(M))$ of M lies above a direct summand of M.*

Proof. This is straightforward as for every $\varphi \in End_R(M)$, $\sum_{\phi\in I}\phi(\overline{Z}^2(M)) = \varphi(\overline{Z}^2(M))$. □

Recall that a module M satisfies $SSSP$ in case the sum of each family of direct summands of M, is a direct summand of M.

Theorem 2.7. *Let M be a module with $SSSP$ on direct summands of M contained in $\overline{Z}^2(M)$. Then M is wTd-Rickart if and only if for every finitely generated right ideal I of S, the submodule $\sum_{\varphi\in I}\varphi(\overline{Z}^2(M))$ of M lies above a direct summand of M.*

Proof. (⇒) Let M be wTd-Rickart and $I =< f_1, \dots, f_n >$ a finitely generated right ideal of $End_R(M)$. It is easy to check that $\sum_{f\in I} f(\overline{Z}^2(M)) = f_1(\overline{Z}^2(M)) + \dots + f_n(\overline{Z}^2(M)) \subseteq \overline{Z}^2(M)$. Set $f = f_1 + \dots + f_n$. Then $f(\overline{Z}^2(M)) = \sum_{\varphi\in I}\varphi(\overline{Z}^2(M))$. Since M is wTd-Rickart, there exist direct summands $D_1, \dots, D_n$ of M such that $\frac{f_1(\overline{Z}^2(M))}{D_1} \ll \frac{M}{D_1}, \dots, \frac{f_n(\overline{Z}^2(M))}{D_n} \ll \frac{M}{D_n}$. Define $D = D_1 + \dots + D_n$. It is not hard to verify that $\frac{f_1(\overline{Z}^2(M))+D}{D} \ll \frac{M}{D}, \dots, \frac{f_n(\overline{Z}^2(M))+D}{D} \ll \frac{M}{D}$. Since a sum of finite small submodules of a module is small in that module, we conclude that $\frac{f(\overline{Z}^2(M))}{D} = \frac{f_1(\overline{Z}^2(M))+D}{D} + \dots + \frac{f_n(\overline{Z}^2(M))+D}{D} \ll \frac{M}{D}$. Note that by assumption D is a direct summand of M.

(⇐) Let $f \in S$. Consider the cyclic right ideal $I =< f >$ of S. By assumption there is a direct summand D of M such that $\frac{\sum_{\varphi\in I}\varphi(\overline{Z}^2(M))}{D} \ll \frac{M}{D}$. Since $f(\overline{Z}^2(M) = \sum_{\varphi\in I}\varphi(\overline{Z}^2(M))$, the result follows.

□

A module M such that all its noncosingular submodules are direct summand, is said to be NS-module (see [15]). Via the concept NS-modules, we remove the condition $SSSP$ in Theorem2.7.

Corollary 2.8. *Let M be a NS-module. Then M is wTd-Rickart if and only if for every finitely generated right ideal I of S, the submodule $\sum_{\varphi\in I}\varphi(\overline{Z}^2(M))$ of M lies above a direct summand of M.*

Proof. Note that any direct summand of M contained in $\overline{Z}^2(M)$ is noncosingular. Now, the result follows from Theorem 2.7 and the fact that any finite sum of noncosingular modules is noncosingular. □

Proposition 2.9. *Let M be a Noetherian module such that for every direct summand D of M and every $f \in End(D)$, $f(\overline{Z}^2(D))$ is nonsmall in D. Then $M = \bigoplus_{i=1}^n M_i$ where each M_i is indecomposable wTd-Rickart such that $End(M_i)$ is a division ring.*

Proof. The result follows from [1, Proposition 2.3] and Proposition 2.5. □

Recall that an R-module M is called *direct projective* if, given any direct summand D of M and every epimorphism $f : M \to D$, there exists an endomorphism ϕ of M such that $\alpha \circ \phi = \pi$ where $\pi : M \to D$ is projection map. We define a set for M as $\mathcal{FZ} = \{f \in S \mid f(\overline{Z}^2(M)) \ll M\}$. It is not hard to check that $\mathcal{FZ}$ is an ideal of S.

Proposition 2.10. *Let M be a direct projective wTd-Rickart module. Then $\frac{S}{\mathcal{FZ}}$ is a regular ring.*

Proof. The proof is exactly is the same as the proof of the first part of [1, Proposition 2.4]. □

Remark 2.11. (1) Let M be a module such that $\overline{Z}^2(M)$ is noncosingular. Then M is T-dual Rickart if and only if M is wTd-Rickart. To prove, we should note that if $\overline{Z}^2(M)$ is noncosingular, then for every $f \in End(\overline{Z}^2(M))$, we have Imf is noncosingular. In particular, for every amply supplemented module M, the concepts T-dual Rickart and wTd-Rickart coincide.

(2) Let M be a module with $Rad(M) = 0$. Then M is wTd-Rickart if and only if M is T-dual Rickart. For if, M is wTd-Rickart, then for every $f \in End(M)$, there is a decomposition $M = D \oplus D'$ such that $D \subseteq f(\overline{Z}^2(M))$ and $D' \cap f(\overline{Z}^2(M)) \ll D'$. Since $Rad(M) = 0$, then $D' \cap f(\overline{Z}^2(M)) = 0$. it follows that $f(\overline{Z}^2(M)) = D$ is a direct summand of M.

Proposition 2.12. *Let M be a noncosingular module. Then the following coincide:*

(1) *M is $\mathcal{I}$-lifting;*
(2) *M is dual Rickart;*
(3) *M is T-dual Rickart;*
(4) *M is wTd-Rickart.*

Proof. (1) ⇒ (2) Let M be $\mathcal{I}$-lifting and $f \in S = End_R(M)$. Then by assumption there is a direct summand D of M contained in Imf such that $\frac{Imf}{D} \ll \frac{M}{D}$. Since M is noncosingular, then Imf and so $\frac{Imf}{D}$ are noncosingular. Now it follows that $Imf = D$ is a direct summand of M.

(2) ⇒ (3) and (3) ⇒ (4) are obvious by definitions.

(4) ⇒ (1) Let $f \in S$. Since M is noncosingular and wTd-Rickart, there is a direct summand D of M such that $\frac{f(\overline{Z}^2(M))}{D} = \frac{f(M)}{D} \ll \frac{M}{D}$. It follows that M is $\mathcal{I}$-lifting.

□

As we expected before, wTd-Rickart property can be inherited by direct summands.

Proposition 2.13. *Every direct summand of an wTd-Rickart module is wTd-Rickart.*

Proof. Let N be a direct summand of M and $M = N \oplus N'$ for $N' \leq M$. Suppose that $f \in End_R(N)$. Then $g = jofo\pi : M \to M$ is a R-homomorphism where $j : N \to M$ is the inclusion and $\pi : M \to N$ is the canonical projection. Since M is wTd-Rickart there is a direct summand K of M such that $\frac{g(\overline{Z}^2(M))}{K} \ll \frac{M}{K}$. From $g(\overline{Z}^2(M)) = f(\overline{Z}^2(N))$ we get that $\frac{f(\overline{Z}^2(N))}{K} \ll \frac{M}{K}$. We shall prove that $\frac{f(\overline{Z}^2(N))}{K} \ll \frac{N}{K}$. So that let $\frac{f(\overline{Z}^2(N))}{K} + \frac{T}{K} = \frac{N}{K}$. It follows that $f(\overline{Z}^2(N)) + T = N$. Therefore, $\frac{f(\overline{Z}^2(M))}{K} + \frac{T+N'}{K} = \frac{M}{K}$. Now by assumption $T + N' = M$. Since $T \subseteq N$, modular law implies that $T = N$. This completes the proof. □

A module M is t-dual Baer in case for every right ideal I of $S = End_R(M)$, the submodule $\sum_{f\in I} f(\overline{Z}^2(M))$ is a direct summand of M ([2]).

Definition 2.14. We call a module M is *weak T-dual Baer*, provided that for every right ideal I of $S = End_R(M)$, the submodule $\sum_{f\in I} f(\overline{Z}^2(M))$ of M lies above a direct summand D of M.

It can be easily seen that for a noncosingular module, the concepts weak T-dual Baer, t-dual Baer and dual Baer coincide. We shall recall that the definition of a $\mathcal{T}$-noncosingular module. A module M is $\mathcal{T}$-noncosingular provided the image of each nonzero endomorphism of M is non-small in M. In other words, $Imf \ll M$ where $f \in End_R(M)$ implies $Imf = 0$ ([7]). Also we should note that if M is $\mathcal{T}$-noncosingular, then M is weak T-dual Baer if and only if M is t-dual Baer.

Theorem 2.15. *Let M be a module such that $\overline{Z}^2(M) \leq_\oplus M$. Then the following statements are equivalent:*

(i) M is weak T-dual Baer;

(ii) For every right ideal J of $End_R(\overline{Z}^2(M))$, there is a direct summand D of $\overline{Z}^2(M)$ such that $\frac{\sum_{f\in I} f(\overline{Z}^2(M))}{D} \ll \frac{\overline{Z}^2(M)}{D}$.

(iii) For every subset A of $B = \{f \in End(M) \mid Imf \subseteq \overline{Z}^2(M)\}$, there is a direct summand D of M such that $\frac{\sum_{h\in A} Imh}{D} \ll \frac{M}{D}$ and for every $\varphi \in End(M)$, there is a direct summand K of M such that $\frac{\varphi(\overline{Z}^2(M))}{K} \ll \frac{M}{K}$.

(iiii) For every $A' \subseteq End_R(M)$, there is a direct summand D of M such that $\frac{\sum_{f\in A'} f(\overline{Z}^2(M))}{D} \ll \frac{M}{D}$.

An example including a wTd-Rickart module which is not weak T-dual Baer, will be provided below.

Example 2.16. Let $R = \prod_{i=1}^{\infty} F_i$ where $F_i = F$ is a field. It is known that R is von Nuemann regular which is not semisimple. By [9, Remark 2.9], R_R is dual Rickart. So R_R is wTd-Rickart since R is an V-ring. Because R is not a semisimple ring, it is not a dual Baer module (see [8, Corollary 2.7]). Note that since R_R is noncosingular, R_R is not (weak) t-dual Baer.

The following is a characterization of hereditary rings in terms of wTd-Rickart modules (see [9, Theorem 2.29])

Proposition 2.17. *The following statements are equivalent for a ring R.*
(1) *R is right hereditary;*
(2) *Every injective right R-module is noncosingular and wTd-Rickart.*

Lemma 2.18. *The following are equivalent for a ring R:*
(1) *Every free right R-module is wTd-Rickart;*
(2) *Every projective right R-module is wTd-Rickart.*

Proof. (1) ⇒ (2) Let M be a projective R-module. Then there is a free R-module F, such that $F = M \oplus K$. Since F is wTd-Rickart, M is also wTd-Rickart by Proposition 2.13.
(2) ⇒ (1) is clear. □

It will be proven that over a right (semi)perfect ring, every (finitely generated) right R-module is wTd-Rickart if and only if every (finitely generated) R-module is weak t-dual Baer.

Theorem 2.19. *Let R be a right perfect (semiperfect) ring. Then the following are equivalent:*
(1) *Every (finitely generated) R-module is weak t-dual Baer;*
(2) *Every (finitely generated) R-module is wTd-Rickart;*
(3) *Every (finitely generated) R-module is T-dual Rickart;*
(4) *Every (finitely generated) noncosingular R-module is I-lifting and for every (finitely generated) R-module M, $\overline{Z}^2(M) \leq_\oplus M$.*

Proof. (1) ⇒ (2) Let M be an R-module and $f \in End(M)$. Since M is weak t-dual Baer, there is a direct summand D of M such that $\frac{\sum_{\varphi\in I}\varphi(\overline{Z}^2(M))}{D} \ll \frac{M}{D}$ where $I =< f >$. It is clear that $\sum_{\varphi\in I}\varphi(\overline{Z}^2(M)) = f(\overline{Z}^2(M))$. So $\frac{f(\overline{Z}^2(M))}{D} \ll \frac{M}{D}$.

(2) ⇒ (3) Let M be an R-module and $f \in End_R(M)$. Since M is wTd-Rickart, there is a direct summand K of M such that $\frac{f(\overline{Z}^2(M))}{D} \ll \frac{M}{D}$. Being M amply supplemented implies that $\overline{Z}^2(M)$ and hence $f(\overline{Z}^2(M))$ is noncosingular. It follows that $f(\overline{Z}^2(M)) = D$ is a direct summand of M.

(3) ⇒ (4) Let M be noncosingular. Then M is T-dual Rickart if and only if M is I-lifting (Proposition 2.12). The second part of (4) follows from [6, Theorem 3.2].

(4) ⇒ (1) Since $\overline{Z}^2(M)$ is a direct summand, it suffices to show that $\overline{Z}^2(M)$ is I-lifting (see Theorem 2.15). This follows from (4), since for every amply supplemented module M, the submodule $\overline{Z}^2(M)$ is noncosingular by [**?** , Corollary 3.4]. □

Corollary 2.20. *Let R be a right perfect (semiperfect) ring. Then the following are equivalent:*

(1) *Every (finitely generated) R-module is weak t-dual Baer;*
(2) *Every (finitely generated) R-module is wTd-Rickart;*
(3) *Every (finitely generated) R-module is T-dual Rickart;*
(4) *Every (finitely generated) R-module is t-dual Baer.*

Proof. The equivalences follows from [6, Theorem 3.20] and Theorem 2.19. □

We shall introduce the concept of relative wTd-Rickart to study finite direct sums of wTd-Rickart modules.

Definition 2.21. Let M and N be two modules. We call M, wTd-Rickart relative to N, in case for every $f : M \to N$, the submodule $f(\overline{Z}^2(M))$ lies above a direct summand of N.

It is clear that M is wTd-Rickart if and only if M is wTd-Rickart relative to M.

Proposition 2.22. *Let M and N be two modules. Then M is wTd-Rickart relative to N if and only if for every direct summand L of M and every supplement submodule K of M, L is wTd-Rickart relative to K.*

Proof. Let $L = e(M)$ where $e^2 = e \in End(M)$. Suppose that M is wTd-Rickart relative to N and for a submodule T of N we have $K + T = N$ and $K \cap T \ll K$. Let $f : L \to K$ be a homomorphism. Since $fe : M \to N$, there is a direct summand D of N such that $\frac{fe(\overline{Z}^2(M))}{D} \ll \frac{N}{D}$. Set $L \oplus L' = M$. So $\overline{Z}^2(L) \oplus \overline{Z}^2(L') = \overline{Z}^2(M)$. Then $e(\overline{Z}^2(M)) = e(\overline{Z}^2(L) \oplus \overline{Z}^2(L')) = e(\overline{Z}^2(L)) = \overline{Z}^2(L)$. Since $fe(\overline{Z}^2(M)) = f(\overline{Z}^2(L))$, then $\frac{f(\overline{Z}^2(L))}{D} \ll \frac{N}{D}$. We shall prove that $\frac{f(\overline{Z}^2(L))}{D} \ll \frac{K}{D}$. To show that, let $\frac{f(\overline{Z}^2(L))}{D} + \frac{B}{D} = \frac{K}{D}$ for $D \subseteq B \subseteq K$. Since $K + T = N$, we have $\frac{f(\overline{Z}^2(L))}{D} + \frac{B+T}{D} = \frac{N}{D}$. Then $B + T = N$. By modular law, $B + (T \cap K) = K$. Now $T \cap K \ll K$ implies that $B = K$. This completes the proof. The converse is obvious since N is a supplement of zero submodule in N. □

Corollary 2.23. *Let M and N be two modules. Then M is wTd-Rickart if and only if for each two direct summands L and K of M, L is wTd-Rickart relative to K.*

Proposition 2.24. *Let $M_1, \dots, M_n, N$ be modules. If N has SSP for direct summands of N contained in $\overline{Z}^2(N)$, then $\bigoplus_{i=1}^n M_i$ is wTd-Rickart relative to N if and only if for each $i = 1, \dots, n$, M_i is wTd-Rickart relative to N.*

Proof. Let $\bigoplus_{i=1}^n M_i$ be wTd-Rickart relative to N. Then by Proposition 2.22, for every $i = 1, \dots, n$, M_i is wTd-Rickart relative to N. For the converse, let $f : \bigoplus_{i=1}^n M_i \to N$. then $f = \oplus_{i=1}^n f_i$ where $f_i : M_i \to N$ for each $i = 1, \dots, n$. Since for each $i = 1, \dots, n$, M_i is wTd-Rickart relative to N, there is a direct summand D_i of N such that $\frac{f_i(\overline{Z}^2(M_i))}{D_i} \ll \frac{N}{D_i}$. Since N has SSP for direct summands of N contained in $\overline{Z}^2(N)$, we get that $D := D_1 + \ldots + D_n$ is a direct summand of N. We shall prove that $\frac{f(\overline{Z}^2(\bigoplus_{i=1}^n M_i))}{D} \ll \frac{N}{D}$. Since $\frac{f_i(\overline{Z}^2(M_i))}{D_i} \ll \frac{N}{D_i}$, it is easy to check that $\frac{f_i(\overline{Z}^2(M_i))+D}{D} \ll \frac{N}{D}$. Now $\sum_{i=1}^n [\frac{f_i(\overline{Z}^2(M_i))+D}{D} \ll \frac{N}{D}$. It follows that $\frac{f(\overline{Z}^2(\bigoplus_{i=1}^n M_i))}{D} \ll \frac{N}{D}$. This completes the proof. □

Corollary 2.25. *Let $M_1, \ldots, M_n$ be modules. Then $\bigoplus_{i=1}^n M_i$ is wTd-Rickart relative to M_j for each $j = 1, \ldots, n$ if and only if each M_i is wTd-Rickart relative to each M_j.*

Acknowledgements:

The authors are very grateful to the referees for many valuable comments and suggestions which helped to improve the paper.

References

[1] T. Amoozgar, *A generalization of lifting modules*, Ukrainian Math. J. **66**(11) (2014), 1654–1664. 1, 2, 2, 2
[2] T. Amouzegar, D. Keskin and Y. Talebi, *t-Dual Baer modules and t-lifting modules*, Vietnam J. Math. **42** (2014), 159–169. 2
[3] T. Amouzegar and A. R. Moniri Hamzekolaee, *Lifting modules with respect to imgaes of a fully invariant submodule*, Novi Sad J. Math. **50**(2) (2020), 41–50. 1
[4] S. Asgari, Y. Talebi and A. R. Moniri Hamzekolaee, *Dual Rickart (Baer) modules and preradicals*, J. Algebraic Syst. **12**(1) (2024), 179–191. 1
[5] J. Clark, C. Lomp, N. Vanaja and R. Wisbauer, *Lifting Modules. Supplements and Projectivity in Module Theory*, Frontiers in Mathematics, Boston, Birkhäuser, (2006). 1
[6] Sh. Ebrahimi Atani, M. Khoramdel and S. Dolati Pish Hesri, *T-dual Rickart modules*, Bull. Iranian Math. Soc. **42**(3) (2016), 627–642. 2, 2.2, 2, 2
[7] D. Keskin and R. Tribak, *On $\mathcal{T}$-noncosingular modules*, Bull. Aust. Math. Soc. **80** (2009), 462–471. 2
[8] D. Keskin and R. Tribak, *On dual Baer modules*, Glasgow J. Math. **52** (2010), 261–269. 2.16
[9] G. Lee, S. T. Rizvi and C. S. Roman, *Dual Rickart modules*, Comm. Algebra **39** (2011), 4036–4058. 1, 2.2, 2.16, 2
[10] S. H. Mohamed and B. J. Müller, *Continuous and Discrete Modules*, London Math. Soc. Lecture Notes Series 147, Cambridge, University Press, (1990). 1
[11] A. R. Moniri Hamzekolaee, *H^*-condition on the set of submodules of a module*, J. Algebraic Struc. Appl. **6**(2) (2019), 13–20. 1
[12] A. R. Moniri Hamzekolaee, *H-supplemented modules and singularity*, J. Algebraic Struc. Appl. **7**(1) (2020), 49–57. 1
[13] A. R. Moniri Hamzekolaee and T. Amouzegar, *H-supplemented modules with respect to images of a fully invariant submodule*, Proyecciones J. Math. **4**(1) (2021), 33–46. 1
[14] A. R. Moniri Hamzekolaee, A. Harmanci, Y. Talebi and B. Ungor, *A new approach to H-supplemented modules via homomorphisms*, Turkish J. Math. **42**(4) (2018), 1941–1955. 1
[15] Y. Talebi, M. Hosseinpour and A. R. Moniri Hamzekolaee, *Modules for which every non-cosingular submodule is a summand*, Bull. Iranian Math. Soc. **43**(3) (2017), 911–922. 2
[16] Y. Talebi, R. Tribak and A. R. Moniri Hamzekolaee, *On H-cofinitely supplemented modules*, Bull. Iranian Math. Soc.**39**(2) (2013), 325–346. 1
[17] Y. Talebi and N. Vanaja, *The torsion theory cogenerated by M-small modules*, Comm. Algebra **30**(3)(2002), 1449–1460. 1
[18] R. Tribak, *On weak dual Rickart modules and dual Baer modules*, Comm. Algebra, **43**(8) (2015), 3190–3206. 1