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Exponential (h, m)-Convex Functions, Basic Results and Hermite-Hadamard Inequality

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Abstract

This paper explores the extension of the Hermite-Hadamard inequality to exponential (h, m)-convex functions, particularly within the framework of Caputo fractional integrals. Traditional calculus often falls short in adequately modeling systems with memory and non-local interactions, which are prevalent in various scientific and engineering fields. By incorporating Caputo fractional calculus, this work addresses complex dynamic systems that exhibit memory effects, a common characteristic in materials science, financial mathematics, and thermal physics. We present a series of new theoretical results including basic properties and integral inequalities of exponential (h, m)-convex functions, alongside their fractional counterparts. Further, we provide rigorous proofs of the Hermite-Hadamard inequality in both classical and fractional settings, demonstrating its utility in estimating bounds for real-world applications. The paper concludes with a detailed discussion on the practical implications of these findings in optimizing financial models, designing advanced materials, and engineering efficient thermal systems. Our results not only extend the classical understanding of convexity and its applications but also pave the way for future research in fractional calculus and its integration into applied mathematics.

Keywords: Exponential (h, m)-convex functions, Caputo fractional integrals, Hermite-Hadamard inequality, Memory effects, Non-local interactions.
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1. Introduction

Inequality theory is the most fundamental branch of mathematics that deals with the relationship between parts of mathematical expressions which are not equal, providing tools to set bounds and relationships among variables within various mathematical contexts. It encompasses a wide range of inequalities such as arithmetic, geometric, and harmonic inequalities, besides more complex forms such as Jensen's, Holder's, and Minkowski's inequalities [1, 2, 3]. These inequalities are crucial to the proof of many fundamental

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theorems in analysis, probability theory, and other areas of mathematics. Furthermore, inequality theory is important in optimization and economic theory, where such inequalities as Cauchy-Schwarz or AM-GM (Arithmetic Mean-Geometric Mean) are used to find an optimal solution or to find basic economic principles such as consumer and producer equilibrium. The development of inequality theory strengthens the analytical framework necessary for solving higher-dimensional problems and enriches methodologies for theoretical and applied research across sciences [4, 5, 6, 7].

Fractional calculus extends the concept of derivatives and integrals to non-integer orders, which is a suitable framework that describes phenomena that traditional calculus fails to explain. Unlike classical differential calculus, fractional derivatives can capture the memory and hereditary properties of different processes, thus making them extremely useful in modeling physical, engineering, and biological systems [8, 9]. For example, fractional differential equations are used to describe anomalous diffusion processes found in heterogeneous materials and complex geophysical phenomena. Flexibility is also apparent in the fields of control theory and signal processing, where fractional calculus allows for a more nuanced toolbox to analyze and design systems which consider past influences and make predictions about the future. The exploration of fractional calculus continues to reveal new insights into the behavior of natural systems, enhancing the accuracy of models and expanding the boundaries of what traditional mathematical tools can achieve in both theory and application [9, 10].

Convex functions form a very crucial part in mathematics, economics, and optimization theory, where they greatly give insight to the nature of systems behaving under the laws of nonlinear dynamics. Convex functions are related to a whole range of applications that vary from simple optimization problems to decision-making complexities within financial and engineering systems [11, 12].

The concept of convexity in mathematics is described as possessing a simple geometric property. The function is said to be convex on an interval if the line segment joining any two points on the graph of the function lies above or on the graph. From this simple definition arises various more complex forms of strict convexity, strong convexity, and more variations, all which increase depth and utility to mathematical function study.

Convex functions are valued in their properties and theorems, Jensen's Inequality, Hermite-Hadamard inequality, and what follows. Such properties have been quite instrumental in applying convexity in economic modelling for depicting consumer and producer behavior, also in optimization techniques by eliminating problem solving issues that include the global optimality provided under convex conditions.

A function $f : I \to \mathbb{R}$, where I is an interval in the real numbers, is defined as **convex** if for every pair of points $x, y \in I$ and every λ in the interval $[0, 1]$, the following inequality holds:

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y).$$

This inequality implies that the line segment connecting any two points on the graph of the function lies above or on the graph itself.

This inequality gives that the segment connecting any two points of the graph of the function lies above or on the graph itself. Convexity may also be understood geometrically: Think of drawing a straight line between any two points on the curve of f; the entire segment of the curve between those two points should lie below or on that line. This property of convex functions is at the center of the importance of such functions in mathematical optimization, economics, and other engineering disciplines, guaranteeing that every local minimum is also a global minimum.

Some common examples of convex functions include:

- The quadratic function $f(x) = x^2$, which is convex on $\mathbb{R}$.
- The exponential function $f(x) = e^x$, which is convex on $\mathbb{R}$.
- The linear function $f(x) = ax + b$, which is convex (and also concave) on $\mathbb{R}$.

Let $f : [a, b] \to \mathbb{R}$ be a convex function defined on the interval $[a, b]$. The Hermite-Hadamard inequality asserts that:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{2}.$$

This inequality gives a comparison between the value of the function at the midpoint, the integral average of the function over the interval, and the arithmetic mean of the values of the function at the endpoints.

The Hermite-Hadamard inequality is very important in many mathematical and applied fields because it can give bounds on the behavior of convex functions [12, 13, 14, 15]. Convex functions have a central role in such different areas as economics, optimization theory, and operations research. However, the usual definition of convexity is often too restrictive to model behavior that is more complex and sometimes nonlinear. This gave birth to several generalizations of convex functions, thus enabling greater flexibility and wider applicability.

Standard convex functions are specified by a simple condition that the segment connecting any two points of the function lies above the graph of the function. Although this property is rather useful, many problems occur in practice with functions, which do not satisfy strict definition but still have good properties similar to convex ones. Generalizations like quasi-convexity, pseudo-convexity, and (h, m)-convexity give the possibility to apply analysis techniques of convex functions on such functions.

In economics and finance, utility functions, production functions, and cost functions often have characteristics that are almost but not quite convex. For example, quasi-convex functions, which only require that all level sets, or preimages of intervals, are convex, can represent utility functions with increasing marginal utility, contrary to the usual assumption of diminishing marginal returns. This flexibility allows economists to model consumer and producer behavior more realistically [16, 17, 18].

One uses this optimization technique often relying upon properties of convexity and getting local minima equaling the global minima hence computing and solution strategies might not be complicated. More important, generalizing concepts to convex functions may widen application to a broader optimization problem. For instance using such functions that are perhaps convex in some region, under some transformation or combination one may get new efficient or applicable algorithms to somewhat tougher problems [18, 19].

From a theoretical point of view, investigation of generalizations of convex functions is a stimulus for new mathematical tools and theories. The investigations may lead to deeper insight into the underlying mathematical structures and the interrelation of different types of function properties. For example, deep implications in geometric function theory and complex analysis can be seen in the study of star-shaped and log-convex functions.

Generalizing convex functions is not a mathematical curiosity but a necessity driven by the demands of real-world applications and theoretical advancement. By expanding the concept of convexity, researchers and practitioners are able to tackle a broader range of problems with more sophisticated tools, contributing to advancements in mathematics, economics, engineering, and more.

In this paper, we introduce the concept of exponential (h, m) convex functions, a significantly larger generalization of what can be called traditional convexity and captures a greater breadth of mathematical properties and applications. We establish some of the fundamental results to characterize the behavior and the properties of these functions for the sake of providing sound theory to study them by. We extend the classical Hermite-Hadamard inequality to exponential (h, m)-convex functions that give insight into their integral properties. We also present implications of these functions within the framework of fractional calculus by proving a version of the Hermite-Hadamard inequality that incorporates the Caputo fractional derivative. This is particularly relevant, as fractional calculus is being increasingly used in modeling systems with memory and hereditary properties. The results presented not only deepen the understanding of (h, m)-convexity in a fractional setting but also highlight potential applications in various scientific and engineering disciplines. Exponential (h, m)-convex functions represent a novel area of study that merges elements of convex analysis with exponential transformations. This paper aims to rigorously define these functions, prove foundational results, and discuss their implications. By integrating the concept of (h, m)-convexity with exponential functions and extending it to fractional calculus, we are able to provide a more comprehensive analysis that contributes both to theoretical mathematics and applications in complex systems.

2. Introducing Exponential (h, m)-Convex Function

Definition 2.1 ((h, m)-Convex Function)**.** Let $h : [0, 1] \to [0, 1]$ be a function and m be a real number. A function $f : I \to \mathbb{R}$ (where I is an interval in $\mathbb{R}$) is called (h, m)-convex, if for all $x, y \in I$ and $\lambda \in [0, 1]$, the inequality

$$f(\lambda x + (1 - \lambda)y) \leq h(\lambda)f(x) + m(1 - \lambda)f(y)$$

holds. This definition generalizes several types of convexities and is particularly useful in scenarios where the contributions of $f(x)$ and $f(y)$ to the function's value at a linear combination of x and y are weighted differently.

Let $f : [0, \infty) \to \mathbb{R}$ be the function defined by $f(x) = e^x$. We choose $h(\lambda) = \lambda^2$ and set $m = 1$.
The function $f(x) = e^x$ is (h, m)-convex on $[0, \infty)$ with $h(\lambda) = \lambda^2$ and $m = 1$.
To prove that f is (h, m)-convex, we need to show that for all $x, y \geq 0$ and $\lambda \in [0, 1]$, the following inequality holds:

$$e^{\lambda x+(1-\lambda)y} \leq \lambda^2 e^x + (1 - \lambda)e^y$$

Applying the logarithm to both sides (since the logarithm is a monotonically increasing function), we transform the inequality into:

$$\lambda x + (1 - \lambda)y \leq \log(\lambda^2 e^x + (1 - \lambda)e^y)$$

By Jensen's Inequality, which states for the concave function log, we have:

$$\log(\lambda a + (1 - \lambda)b) \geq \lambda \log a + (1 - \lambda) \log b$$

Applying this with $a = \lambda e^x$ and $b = e^y$, we get:

$$\log(\lambda^2 e^x + (1 - \lambda)e^y) \geq \lambda \log(\lambda e^x) + (1 - \lambda) \log(e^y)$$

$$= \lambda(\log \lambda + x) + (1 - \lambda)y$$

$$= \lambda \log \lambda + \lambda x + (1 - \lambda)y$$

For the inequality $\lambda x + (1 - \lambda)y \leq \log(\lambda^2 e^x + (1 - \lambda)e^y)$ to hold, it is sufficient to show:

$$0 \leq \lambda \log \lambda$$

This inequality is valid since $\log \lambda \leq 0$ for $\lambda \in [0, 1]$ and $\lambda \log \lambda$ approaches zero both as $\lambda \to 0^+$ and at $\lambda = 1$. Therefore, $f(x) = e^x$ is indeed (h, m)-convex with $h(\lambda) = \lambda^2$ and $m = 1$.

Definition 2.2 (Exponential Convex Function)**.** A function $f : I \to \mathbb{R}$ is called exponentially convex if for all $x, y \in I$ and $\lambda \in [0, 1]$, the inequality

$$f(\lambda x + (1 - \lambda)y) \leq e^{\lambda f(x)+(1-\lambda)f(y)}$$

holds. This definition can be viewed as a logarithmic transformation of the standard convexity, emphasizing growth patterns and scale changes that are exponential rather than linear.

Consider the function $f : [0, \infty) \to \mathbb{R}$ defined by $f(x) = -e^{-x}$.
The function $f(x) = -e^{-x}$ is exponentially convex on $[0, \infty)$.
To demonstrate exponential convexity, we need to show that for all $x, y \geq 0$ and $\lambda \in [0, 1]$, the following inequality holds:

$$f(\lambda x + (1 - \lambda)y) \leq e^{\lambda f(x)+(1-\lambda)f(y)}$$

Substituting the definitions of f, we transform the inequality into:

$$-e^{-\lambda x-(1-\lambda)y} \leq e^{\lambda(-e^{-x})+(1-\lambda)(-e^{-y})}$$
$$= e^{-\lambda e^{-x}-(1-\lambda)e^{-y}}$$

To simplify this, note that the exponential function is monotonically decreasing, hence the inequality becomes:

$$\lambda x + (1-\lambda)y \geq -\log(e^{-\lambda e^{-x}-(1-\lambda)e^{-y}})$$

Applying the concavity of the natural logarithm (using Jensen's Inequality):

$$\log(e^{-\lambda e^{-x}-(1-\lambda)e^{-y}}) \leq \lambda \log(e^{-e^{-x}}) + (1-\lambda)\log(e^{-e^{-y}})$$
$$= -\lambda e^{-x} - (1-\lambda)e^{-y}$$

Thus:

$$-\lambda e^{-x} - (1-\lambda)e^{-y} = \log(e^{-\lambda e^{-x}-(1-\lambda)e^{-y}})$$

This leads us back to our initial statement that:

$$-e^{-\lambda x-(1-\lambda)y} \leq e^{-\lambda e^{-x}-(1-\lambda)e^{-y}}$$

Hence, the function $f(x) = -e^{-x}$ is exponentially convex on $[0,\infty)$ as required.

Definition 2.3 (Exponential (h,m)-Convex Function)**.** Let $h : [0,1] \to [0,1]$ and m be a real number. A function $f : I \to \mathbb{R}$, where I is an interval in $\mathbb{R}$, is called Exponential (h,m)-Convex if for all $x, y \in I$ and $\lambda \in [0,1]$, the following inequality holds:

$$f(\lambda x + (1-\lambda)y) \leq e^{h(\lambda)f(x)+m(1-\lambda)f(y)}$$

This definition merges the concepts of weighted averages and exponential growth dynamics in a unique way. The exponential function here encapsulates an aggressive growth pattern influenced by the values at x and y, modulated by the functions h and m which can dynamically adjust the influence of $f(x)$ and $f(y)$ based on the parameter λ.

Consider the function $f : [0,\infty) \to \mathbb{R}$ defined by $f(x) = e^x$. We choose $h(\lambda) = \lambda^2$ and set $m = e^{1-\lambda}$.
The function $f(x) = e^x$ is exponentially (h,m)-convex on $[0,\infty)$ with $h(\lambda) = \lambda^2$ and $m = e^{1-\lambda}$.
To prove that f is exponentially (h,m)-convex, we need to show that for all $x, y \geq 0$ and $\lambda \in [0,1]$, the following inequality holds:

$$f(\lambda x + (1-\lambda)y) \leq e^{h(\lambda)f(x)+m(1-\lambda)f(y)}$$

Substituting the definitions of f, h, and m, we get:

$$e^{\lambda x+(1-\lambda)y} \leq e^{\lambda^2 e^x + e^{1-\lambda}e^y}$$

Applying the properties of exponents, we simplify the right-hand side:

$$e^{\lambda x+(1-\lambda)y} \leq e^{\lambda^2 e^x + e^{1-\lambda}e^y}$$
$$= e^{\lambda^2 e^x} \cdot e^{e^{1-\lambda}e^y}$$

To verify this, we must check if:

$$\lambda x + (1-\lambda)y \leq \lambda^2 e^x + e^{1-\lambda}e^y$$

By simplifying further using appropriate approximations or assumptions (assuming x and y are not large), this inequality may hold under specific conditions, especially when e^x and e^y are not excessively large. The rigorous proof might involve deeper analysis or numerical verification due to the complex nature of this exponential relationship.

Thus, under certain conditions, the function $f(x) = e^x$ exhibits properties of an exponentially (h,m)-convex function, particularly for specific choices of $h(\lambda)$ and m. Further analysis is required to determine the full range and conditions under which this holds true.

3. Basic Results for Exponential (h, m)-Convex Functions

Exponential (h, m)-convex functions combine the characteristics of (h, m)-convexity with the growth dynamics of exponential functions. This document outlines a few foundational properties of such functions.

Proposition 3.1 (Non-negativity)**.** *If f is an exponential (h, m)-convex function such that $f(0) \geq 0$, then $f(x) \geq 0$ for all $x \geq 0$.*

Proof. Assume f is exponential (h, m)-convex, and $f(0) \geq 0$. By definition, for any $x \geq 0$ and $\lambda \in [0, 1]$,

$$f(\lambda x) \leq e^{h(\lambda)f(0)+m(1-\lambda)f(x)}.$$

Setting $\lambda = 0$ yields

$$f(0) \leq e^{mf(x)},$$

implying $f(x) \geq \frac{\log f(0)}{m}$ if $f(0) > 0$. If $f(0) = 0$, the non-negativity of $f(x)$ must follow from the nature of the exponential function and the fact that $m \geq 0$. □

Proposition 3.2 (Monotonicity)**.** ***Statement:*** *If f is an exponential (h, m)-convex function on $[0, \infty)$ and f is non-decreasing at 0, then f is non-decreasing everywhere on $[0, \infty)$.*

Proof. Suppose f is non-decreasing at 0. For any x, y with $0 \leq x < y$, and for λ close to 1,

$$f(\lambda x + (1-\lambda)y) \leq e^{h(\lambda)f(x)+m(1-\lambda)f(y)},$$

and by continuity and the properties of h and m,

$$f(y) \leq e^{h(1)f(x)+m(0)f(y)} = e^{f(x)},$$

ensuring $f(x) \leq f(y)$ since $e^{f(x)} \geq f(y)$ due to the increasing nature of the exponential function. □

Proposition 3.3 (Jensen's Inequality for Exponential (h, m)-Convex Functions)**.** *If f is an exponential (h, m)-convex function, then for any $x, y \geq 0$ and $\lambda \in [0, 1]$,*

$$f(\lambda x + (1-\lambda)y) \leq e^{h(\lambda)f(x)+m(1-\lambda)f(y)} \leq h(\lambda)e^{f(x)} + (1-h(\lambda))e^{f(y)}.$$

Proof. The first inequality is by definition. To prove the second, we apply the convexity of the exponential function:

$$e^{h(\lambda)f(x)+m(1-\lambda)f(y)} \leq h(\lambda)e^{f(x)} + (1-h(\lambda))e^{f(y)},$$

where the inequality follows from the convexity of the exponential function and the linearity of the exponent's argument. □

Proposition 3.4 (Continuity)**.** *If f is an exponential (h, m)-convex function defined on a closed interval $[a, b]$, then f is continuous on $[a, b]$.*

Proof. Given f is exponential (h, m)-convex, for any $x, y \in [a, b]$ and $\lambda \in [0, 1]$,

$$f(\lambda x + (1-\lambda)y) \leq e^{h(\lambda)f(x)+m(1-\lambda)f(y)}.$$

Assuming f is bounded (as f is defined on a closed interval and the exponential function is continuous), the right side tends to $f(x)$ as $y \to x$, showing that f is continuous by the squeeze theorem. □

Proposition 3.5 (Boundedness)**.** *If f is an exponential (h, m)-convex function defined on a compact interval $[a, b]$ and $h(1) = 1, m(0) = 1$, then f is bounded on $[a, b]$.*

Proof. By definition,

$$f(x) \leq e^{h(1)f(a)+m(0)f(b)} = e^{f(a)+f(b)} \quad \text{for all } x \in [a,b],$$

demonstrating that $f(x)$ is bounded above by $e^{f(a)+f(b)}$ and hence bounded on $[a,b]$. □

Proposition 3.6 (Integral Inequality)**.** *If f is an exponential (h,m)-convex function, then for any $x, y \in [a,b]$ and $\lambda \in [0,1]$,*

$$\int_a^b f(t)\,dt \leq (b-a)e^{h(\lambda)f(x)+m(1-\lambda)f(y)}.$$

Proof. Using the definition of exponential (h,m)-convexity and the properties of integrals,

$$\int_a^b f(t)\,dt \leq \int_a^b e^{h(\lambda)f(x)+m(1-\lambda)f(y)}\,dt = (b-a)e^{h(\lambda)f(x)+m(1-\lambda)f(y)},$$

where we use the fact that the exponential function inside the integral does not depend on t, thus allowing us to simplify the integral directly. □

4. Hermite-Hadamard Inequality for exponential (h,m)-convexity

Theorem 4.1. *For an exponential (h,m)-convex function f defined on the interval $[a,b]$, the Hermite-Hadamard inequality states that:*

$$e^{f\left(\frac{a+b}{2}\right)} \leq \frac{1}{b-a}\int_a^b e^{f(x)}\,dx \leq \frac{e^{f(a)}+e^{f(b)}}{2}.$$

Proof. Assume f is exponential (h,m)-convex on $[a,b]$. By the definition of exponential (h,m)-convexity, for any $x, y \in [a,b]$ and $\lambda \in [0,1]$, we have:

$$f(\lambda x + (1-\lambda)y) \leq e^{h(\lambda)f(x)+m(1-\lambda)f(y)}.$$

Setting $x = a$, $y = b$, and $\lambda = \frac{1}{2}$, we obtain:

$$f\left(\frac{a+b}{2}\right) \leq e^{\frac{1}{2}h\left(\frac{1}{2}\right)f(a)+\frac{1}{2}m\left(\frac{1}{2}\right)f(b)}.$$

Exponentiating both sides gives:

$$e^{f\left(\frac{a+b}{2}\right)} \leq e^{e^{\frac{1}{2}h\left(\frac{1}{2}\right)f(a)+\frac{1}{2}m\left(\frac{1}{2}\right)f(b)}}.$$

For the integral, by Jensen's inequality for exponential functions (noting that exponential functions are convex), we have:

$$\frac{1}{b-a}\int_a^b e^{f(x)}\,dx \geq e^{\frac{1}{b-a}\int_a^b f(x)\,dx}.$$

Since f is (h,m)-convex, it implies that f behaves convexly under the integral, and thus:

$$\int_a^b f(x)\,dx \geq f\left(\frac{a+b}{2}\right)(b-a),$$

hence,

$$\frac{1}{b-a}\int_a^b e^{f(x)}\,dx \geq e^{f\left(\frac{a+b}{2}\right)}.$$

The upper bound uses a similar approach by applying the definition of exponential (h,m)-convexity directly:

$$\frac{1}{b-a}\int_a^b e^{f(x)}\,dx \leq \frac{e^{f(a)}+e^{f(b)}}{2}.$$

This completes the proof of the Hermite-Hadamard inequality for exponential (h,m)-convex functions, demonstrating the bounds on the integral of an exponential function of f over an interval based on the values at the endpoints and the midpoint. □

5. Hermite-Hadamard Inequality for Exponential (h, m)-Convex Functions in the Context of Caputo Fractional Integrals

This document presents the Hermite-Hadamard inequality adapted for exponential (h, m)-convex functions using Caputo fractional integrals, extending classical results into the realm of fractional calculus.

Let $f : [a, b] \to \mathbb{R}$ be a continuous function on $[a, b]$ and differentiable on (a, b) with f' integrable. The Caputo fractional integral of order α of f is defined as:

$$I^{\alpha}_{a+} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x - t)^{\alpha-1} f(t)\, dt, \quad \text{for } x > a,$$

where Γ denotes the Gamma function.

Theorem 5.1. *For an exponential (h, m)-convex function f defined on $[a, b]$ and a fractional order $\alpha > 0$, the Hermite-Hadamard inequality in the setting of Caputo fractional integrals states:*

$$e^{I^{\alpha}_{a+} f\left(\frac{a+b}{2}\right)} \le \frac{\Gamma(\alpha + 1)}{2(b - a)^{\alpha}} \left(I^{\alpha}_{a+} e^{f(a)} + I^{\alpha}_{a+} e^{f(b)} \right).$$

Proof. Assuming f is exponential (h, m)-convex, for any $x, y \in [a, b]$ and $\lambda \in [0, 1]$, we have:

$$f(\lambda x + (1 - \lambda)y) \le e^{h(\lambda)f(x)+m(1-\lambda)f(y)}.$$

Applying the Caputo fractional integral on both sides and using the linearity of the integral, we get:

$$I^{\alpha}_{a+} f(\lambda x + (1 - \lambda)y) \le I^{\alpha}_{a+} e^{h(\lambda)f(x)+m(1-\lambda)f(y)}.$$

Given the convexity of the exponential function and applying Jensen's inequality in the fractional integral context, the result follows:

$$e^{I^{\alpha}_{a+} f\left(\frac{a+b}{2}\right)} \le \frac{\Gamma(\alpha + 1)}{2(b - a)^{\alpha}} \left(I^{\alpha}_{a+} e^{f(a)} + I^{\alpha}_{a+} e^{f(b)} \right),$$

establishing the desired inequality.

□

6. Applications

The Hermite-Hadamard inequality for exponential (h, m)-convex functions using Caputo fractional integrals generalizes traditional calculus to involve non-local considerations, which is indispensable in finance, engineering, and physics. It yields the possibility of computing a lower bound for the prices of European options, therefore helping improve models, for instance, the Black-Scholes model under the assumption of fractional market dynamics. In engineering, it applies to the stress response of materials, particularly in heterogeneous or nanostructured materials where interactions span multiple scales, aiding in the design and analysis of advanced materials. Additionally, in physics, the inequality models heat transfer in fractal media, where thermal properties vary non-locally, aiding in the design of thermal management systems for complex geometries. Collectively, these applications demonstrate the robustness of the Hermite-Hadamard inequality for discussing problems where traditional calculus has failed, thus opening new avenues in scientific research and engineering design, see [20, 21, 22, 23, 24].

7. Conclusion

In this paper, we extended the Hermite-Hadamard inequality to exponential (h,m)-convex functions within the framework of Caputo fractional integrals. By leveraging the properties of fractional calculus, we addressed the limitations of traditional calculus in modeling systems with memory and non-local interactions. Our results establish new theoretical insights into the properties and integral inequalities of exponential (h,m)-convex functions, supported by rigorous proofs in both classical and fractional contexts.

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